Applied Mathematics and Computation 265 (2015) 723–732

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Sharp upper and lower bounds for the moments of Bernstein polynomials

José A. Adell[a], Jorge Bustamante[b], José M. Quesada[c,*]

[a] Department of Statistics, Universidad de Zaragoza, Spain
[b] Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, Mexico
[c] Department of Mathematics, Universidad de Jaén, Spain

A R T I C L E I N F O

Keywords:
Bernstein polynomials
Bernstein moments
Best constants

A B S T R A C T

We give upper and lower bounds for the moments and the uniform moments of Bernstein polynomials. Asymptotically, such bounds are best possible.

1. Introduction and main results

Throughout this paper, $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\tau(x) = x(1-x)$, $x \in [0,1]$. We will denote by $\operatorname{sgn}(\cdot)$ the real function defined by $\operatorname{sgn}(z) = 1$, if $z > 0$, $\operatorname{sgn}(z) = -1$, if $z < 0$, and $\operatorname{sgn}(0) = 0$. For any function $f : [0,1] \to \mathbb{R}$ and $n \in \mathbb{N}$, the nth Bernstein polynomial of f is defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}. \quad (1)$$

In many problems concerning the behaviour of Bernstein polynomials, it is necessary to estimate the absolute value of the central moments

$$\mu_{n,k}(x) = B_n((e_1 - x)^k, x), \quad n \in \mathbb{N}, \quad k \in \mathbb{N}_0, \quad (2)$$

where $e_1(t) = t$. Bernstein [2] was the first to note that the central moments are important in obtaining Voronoskaja's type theorems. In this respect, the following inequalities are known

[2, Bernstein],

$$| \mu_{n,k}(x) | \le \frac{k!}{n^{k/2}} \exp(\tau(x)), \quad \max\{| x |, | 1-x |\} \le \sqrt{n},$$

[10, Veselinov],

$$| \mu_{n,k}(x) | \le \frac{k!\, n^{1/5}}{(n \ln n)^{k/2}}, \quad x \in [0,1], \quad n \ge 2,$$

[6, p. 15, Lorentz],

$$\mu_{n,2k}(x) \le \frac{M_k}{n^k}, \quad x \in [0,1],$$

* Corresponding author. Tel.: +34683750029.
E-mail addresses: adell@unizar.es (J.A. Adell), jbusta@fcfm.buap.mx (J. Bustamante), jquesada@ujaen.es, hyh@csu.edu.cn (J.M. Quesada).

http://dx.doi.org/10.1016/j.amc.2015.05.113
0096-3003/

[3, Ditzian],

$$\mu_{n,2k}(x) \leq M_k \left(\frac{x(1-x)}{n} \right)^k, \quad B > 0, \quad \frac{B}{n} \leq x \leq 1 - \frac{B}{n},$$

[4, Gavrea-Ivan]

$$| \mu_{n,k+1}(x) | \leq \frac{M_k x(1-x)}{n^k}, \quad nx(1-x) \leq 1,$$

where $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, and, in each case, M_k is an unspecific constant only depending upon k.

The aim of this paper is to improve the aforementioned estimates by providing explicit upper and lower bounds for the central moments $\mu_{n,k}(x)$, as well as for the corresponding uniform moments $\mu_{n,k}$ defined by

$$\mu_{n,k} = \sup_{0 \leq x \leq 1} |\mu_{n,k}(x)|, \quad n \in \mathbb{N}, \quad k \in \mathbb{N}_0.$$

As we will see, all of such bounds are best possible from an asymptotic point of view.

Unless otherwise specified, we assume from now on that $x \in [0, 1]$. A direct computation shows that (see also [2] and [6, p. 14])

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = 0, \quad \mu_{n,2}(x) = \frac{\tau(x)}{n}, \quad \mu_{n,3}(x) = \frac{(1-2x)\tau(x)}{n^2}, \tag{3}$$

$$\mu_{n,4}(x) = \frac{3\tau(x)^2}{n^2} + \frac{\tau(x)}{n^3}(1 - 6\tau(x)) \tag{4}$$

and

$$\mu_{n,5}(x) = \left(\frac{10\tau(x)^2}{n^3} + \frac{\tau(x)}{n^4}(1 - 12\tau(x)) \right)(1-2x). \tag{5}$$

The main results of this paper are stated as follows.

Theorem 1. *For any $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$\left| \mu_{n,2k+1}(x) - A_k(1-2x)\frac{\tau(x)^k}{n^{k+1}} \right| \leq \frac{B_k \tau(x)}{n^{k+2}} |1-2x| \tag{6}$$

and

$$\left| \mu_{n,2k}(x) - C_k \frac{\tau(x)^k}{n^k} \right| \leq \frac{D_k \tau(x)}{n^{k+1}}, \tag{7}$$

where

$$A_k := \frac{k(2k+1)!}{3\,k!\,2^k}, \quad B_k := (k-1)A_k, \quad C_k := \frac{(2k)!}{k!\,2^k} \quad \text{and } D_k := k(k-1)C_k. \tag{8}$$

Theorem 2. *In the setting of Theorem 1, we have*

$$\left| \mu_{n,2k+1} - \frac{A_k k^k}{2^k(2k+1)^{k+1/2} n^{k+1}} \right| \leq \frac{B_k}{6\sqrt{3} n^{k+2}} \tag{9}$$

and

$$\left| \mu_{n,2k} - \frac{C_k}{4^k n^k} \right| \leq \frac{D_k}{4n^{k+1}}. \tag{10}$$

An immediate consequence of the previous results is the following:

Corollary 1. *For any $k \in \mathbb{N}_0$, we have*

$$\lim_{n\to\infty} n^{k+1}\mu_{n,2k+1}(x) = A_k \tau(x)^k (1-2x), \tag{11}$$

$$\lim_{n\to\infty} n^k \mu_{n,2k}(x) = C_k \tau(x)^k, \tag{12}$$

$$\lim_{n\to\infty} n^{k+1}\mu_{n,2k+1} = \frac{A_k k^k}{2^k(2k+1)^{k+1/2}}, \tag{13}$$

and

$$\lim_{n\to\infty} n^k \mu_{n,2k} = \frac{C_k}{4^k}. \tag{14}$$