Applied Mathematics and Computation 265 (2015) 844–853

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A strongly convergent algorithm for the split common fixed point problem

Oganeditse A. Boikanyo*

Department of Mathematics and Statistics, Botswana International University of Science and Technology, Private Bag 16, Palapye, Botswana

ARTICLE INFO

MSC:
47J25
47H45
65J15

Keywords:
Split common fixed point problem
Demicontractive operator
Quasi-nonexpansive operator
Directed operator

ABSTRACT

Recently, Cui and Wang (2014) constructed an algorithm for demicontractive operators that converges weakly, under mild assumptions, to some solution of the split common fixed point problem. In this paper, based on Halpern's type method (1967), we construct an algorithm for demicontractive operators that produces sequences that always converge strongly to a specific solution of the split common fixed point problem. Particular cases of directed operators and quasi-nonexpansive mappings are also considered.

1. Introduction

The split feasibility problem was first introduced in 1994 by Censor and Elfving [4]. In the setting of a real Hilbert space H, this problem involves finding an element/point in a closed convex subset C of H whose image under a bounded linear operator is an element of another closed and convex subset of H. A particular case of the split feasibility problem is the split common fixed point (SCFP) problem which is an inverse problem that consists of finding an element in a fixed point set whose image under a bounded linear operator is an element of another fixed point set. Since its inception, the split feasibility problem has drawn a lot of interest from several scholars [5,6,10,11,13,16,17] mainly due to its applications in intensity modulated radiation therapy [3], signal processing and image reconstruction [2].

When solving the split feasibility problem, Censor and Elfving [4] introduced an iterative process that involves the computation of an inverse matrix, a task that is not always easy to do. The aim of Byrne's papers [1,2] was mainly to address and overcome this difficulty by introducing an algorithm with a given step size that uses orthogonal projections onto closed and convex sets C and Q. Although the CQ algorithm of Byrne does not involve the computation of the inverse of a matrix, it can only be easily implemented in cases where the projections involved can be computed easily (e.g., C and Q are closed balls or half-spaces). However, if the sets C and Q are arbitrary, but closed and convex such as fixed point sets, the projections into these sets are generally hard to be accurately calculated. This limitation renders the method inefficient in important cases such as these convex sets.

In 2009, Censor and Segal [5] introduced an algorithm with a given step size ρ for finding solutions of the split common fixed point problem. Their algorithm was extended to the case of quasi-nonexpansive operators by Moudafi [10], finitely many directed operators by Wang and Xu [13], and demicontractive mappings by Moudafi [11]. In the case of directed operators, the step size ρ was chosen in such a way that it depends on the norm of the bounded linear operator A, and the algorithm converges weakly to some solution of the SCFP problem. Note that, as mentioned in [6], in order to implement Censor and Segal's algorithm with this choice of step size, one needs to first compute (or, at least, estimate) the norm of A, which is in general not an easy task

* Tel.: +267 4900 117.
E-mail address: boikanyoa@gmail.com, boikanyoa@yahoo.co.uk

http://dx.doi.org/10.1016/j.amc.2015.05.130
0096-3003/

in practice. Cui and Wang's idea [6] was to construct an algorithm whose step size does not depend on the norm of the operator A. Cui and Wang [6] then showed that under appropriate conditions sequences generated by their algorithm converges weakly to a solution of the SCFP problem.

Motivated by Cui and Wang's work [6], we construct an algorithm for demicontractive operators that produces sequences that always converge (under appropriate conditions) strongly to a solution of the SCFP problem and whose step size does not depend on the norm of the operator A. The constructed algorithm is of Halpern type [8]. We also consider particular cases such as quasi-nonexpansive mappings and directed operators.

2. Mathematical background

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H and K, respectively. Let $A: H \to K$ be a bounded linear operator with its adjoint A^*. The split feasibility problem studied by Censor and Elfving [4] consists of finding an element, say x in H, with the property

$$x \in C \quad \text{such that} \quad Ax \in Q. \tag{2.1}$$

To solve this problem, Censor and Elfving [4] introduced the algorithm

$$x_{n+1} = A^{-1}P_Q(P_{A(C)}(Ax_n)), \quad n \in \mathbb{N}, \tag{2.2}$$

for an arbitrary initial guess x_0, where C and Q are closed and convex subsets in $\mathbb{R}^n$, A is a full rank $n \times n$ matrix, $A(C) = \{y \in \mathbb{R}^n | y = Ax, x \in C\}$ and P_Q is the orthogonal projection onto Q. The disadvantage with this algorithm is that it requires the computation of the inverse matrix A^{-1}. We remark that a point x^* in H solves problem (2.1) if and only if x^* is a fixed point of the map $T: H \to C$, where $T := P_C(I - \rho A^*(I - P_Q)A)$ for $\rho > 0$. Recently, Byrne [1,2] introduced the following algorithm that does not involve the computation of the inverse matrix A^{-1}: for an initial guess x_0, a sequence (x_n) is generated recursively by the rule

$$x_{n+1} = P_C(I - \rho A^*(I - P_Q)A)x_n, \quad n \in \mathbb{N}, \tag{2.3}$$

where $\rho \in (0, \frac{2}{L})$ with L taken as the largest eigenvalue of the matrix A^*A. The CQ algorithm (2.3) is also effective in solving problem (2.1) for general real Hilbert spaces H and K, provided that the projections P_C and P_Q are easily calculated.

Another interesting problem that is closely related to the split feasibility problem is the split common fixed point problem and consists of finding an element, say x in H, with the property

$$x \in \mathrm{Fix}(U) \quad \text{such that} \quad Ax \in \mathrm{Fix}(T), \tag{2.4}$$

where $\mathrm{Fix}(U)$ and $\mathrm{Fix}(T)$ are, respectively, the fixed point sets of $U: H \to H$ and $T: K \to K$. This problem was introduced in 2009 by Censor and Segal [5], who invented the following algorithm for solving such a problem: for an arbitrary point x_0, generate a sequence (x_n) recursively by the rule

$$x_{n+1} = U(I - \rho A^*(I - T)A)x_n, \quad n \in \mathbb{N}. \tag{2.5}$$

This algorithm was extended to the case of quasi-nonexpansive operators by Moudafi [10], finitely many directed operators by Wang and Xu [13], and demicontractive mappings by Moudafi [11]. In the case when U and T are directed operators, the step size ρ is chosen in such a way that

$$0 < \rho < \frac{2}{\|A\|^2}$$

and the sequence generated by (2.5) converges weakly to a solution of problem (2.4) whenever such a solution exists. Note that, as mentioned in [6], in order to implement algorithm (2.5) with this choice of step, one needs to first compute (or, at least, estimate) the norm of A, which is in general not an easy task in practice. Cui and Wang [6] constructed an algorithm whose step size does not depend on the operator norm $\|A\|$. More precisely, they introduced the following algorithm for demicontractive operators $U: H \to H$ and $T: K \to K$ with demicontractive constants $\kappa < 1$ and $\tau < 1$, respectively:

Algorithm 2.1. *Choose an initial guess $x_0 \in H$ arbitrarily and $\lambda \in (0, 1 - \tau)$. Assume that the nth iterate x_n has been constructed; then calculate the $(n+1)$th iterate via the formula*

$$x_{n+1} = U_\lambda(x_n - \rho_n A^*(I - T)Ax_n), \quad n \geq 0, \tag{2.6}$$

where A is a bounded linear operator with adjoint A^* and the step size ρ_n is chosen in such a way that

$$\rho_n := \begin{cases} \dfrac{(1-\tau)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n) \\ 0 & otherwise. \end{cases}$$

It was shown [6] that under appropriate conditions the sequence generated by this algorithm converges weakly to a solution x^* of problem (2.4). Motivated by Cui and Wang's work [6], we construct an algorithm for demicontractive operators that converges (under appropriate conditions) strongly to a solution x^* of problem (2.4) and whose step size ρ_n does not depend on the operator norm $\|A\|$. Particular cases such as quasi-nonexpansive mappings and directed operators are also considered.