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Limit periodic homogeneous linear difference systems

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ABSTRACT

We study limit periodic homogeneous linear difference systems, where the coefficient matrices belong to a bounded group. We find groups of matrices with the property that the systems, which do not possess any non-zero asymptotically almost periodic solution, form a dense subset in the space of all considered systems. Analogously, we analyse almost periodic systems as well.

1. Introduction

Let us consider the limit periodic (and almost periodic) homogeneous linear difference systems

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}. \tag{1}$$

Limit periodic systems (1) form the smallest class of systems which generalize pure periodic systems (1) and which can have at least one non-almost periodic solution (for complex matrices A_k from a bounded group). The basic motivation of our research comes from papers [20,41], where we study non-almost periodic solutions of almost periodic systems. In this paper, we improve the main results of [20,41], because we analyse non-almost periodic solutions in the limit periodic case. In addition, we obtain results about non-asymptotically almost periodic solutions. Since the method used in this paper is substantially different from the processes applied in [20,41], we also obtain new results for almost periodic systems. We remark that the treated systems are viewed in the general setting, when the elements of considered sequences are from a field with an absolute value (although we get new results even for the field of complex numbers).

For example, let us point out coefficient matrices A_k from the unitary group. It is proved in [35] (see also [33]) that, in any neighbourhood of an almost periodic unitary system (1), there exist almost periodic unitary systems (1) which possess non-almost periodic solutions. In this paper, we prove direct generalisations of this property of the almost periodic unitary systems. Note that the corresponding result about orthogonal difference, skew-Hermitian differential, and skew-symmetric differential systems is proved in [38,40] (see also [34]), and in [42], respectively. The reason for the study of non-almost periodic solutions lies in the fact that the considered systems have only almost periodic solutions in many cases (for unitary, orthogonal, skew-Hermitian, and skew-symmetric systems, see [23,24,36,38]).

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The necessity of generalisations of periodic mathematical models is implied by various oscillatory phenomena in natural sciences. The models induce the research of limit periodic, almost periodic, and asymptotically almost periodic sequences in connection with difference equations. There are many significant books dealing with (asymptotically) almost periodic solutions of difference and differential equations, e.g., [6,9,12,15,44]. We also refer to the references given in these books.

Now we mention a short literature overview concerning almost periodic difference equations. The basics of the theory of almost periodicity (limit periodicity, asymptotic almost periodicity) can be found in classical books [4,12,15,31,45]. Complex almost periodic systems (1) and their properties (namely exponential dichotomy) are studied in [3,22,30]. Concerning the almost periodicity of solutions of almost periodic linear difference equations, see papers [7,8,11,18,21,46]. Criteria ensuring the existence of almost periodic solutions of general difference systems are proved, e.g., in [19,47,48]. For linear almost periodic equations with $k \in \mathbb{N}$, we refer to [1,32].

In this paper, we use a method based on a construction of limit periodic sequences. Similar method was firstly applied in [16], where non-almost periodic solutions of homogeneous linear difference equations are found as classes of constructible sequences. A method of constructions of minimal cocycles, which are obtained as solutions of homogeneous linear differential systems, is described in [28] (see also [29]). Special constructions of homogeneous linear differential systems with almost periodic coefficients are used in [25,26,40,42] as well.

This paper is organised as follows. We begin with the notation. In Section 3, we recall the concepts of limit periodicity, almost periodicity, and asymptotic almost periodicity and we mention some properties of the considered generalisations of periodicity. In Section 4, we state the notion of (weakly) transformable groups of matrices and we state the strongest known result concerning non-almost periodic solutions of (1), when the coefficient matrices belong to a (weakly) transformable group. Then we introduce properties (denoted by P and P^*) which allow us to improve this result. Our results are formulated and proved in Section 5. We also give several examples.

2. Preliminaries

In this section, we mention the used notation which corresponds to the standard one. Let $(F, \oplus, \odot)$ be an infinite field with a zero e_0 and a unit e_1. Let $|\cdot| : F \to \mathbb{R}$ be an absolute value on F; i.e., let

(a) $|f| \geq 0$ and $|f| = 0$ if and only if $f = e_0$,
(b) $|f \odot g| = |f| \cdot |g|$,
(c) $|f \oplus g| \leq |f| + |g|$

for all $f, g \in F$. As $\mathbb{F}$, we will understand each one of the fields $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ with the usual absolute value. We remark that $\mathrm{i} \in \mathbb{C}$ will stand for the imaginary unit.

Let $m \in \mathbb{N}$ be arbitrarily given (as the dimension of systems under consideration). The symbol $Mat(F, m)$ will denote the set of all $m \times m$ matrices with elements from F and F^m the set of all $m \times 1$ vectors with entries from F. Next, $\cdot, +$ stand for the multiplication and the addition on spaces $Mat(F, m)$, F^m. As usual, we will denote the identity matrix $I \in Mat(F, m)$, the zero matrix $O \in Mat(F, m)$, and the zero vector $o \in F^m$.

The absolute value on F gives the norm on F^m and $Mat(F, m)$ as the sum of m and m^2 non-negative numbers which are the absolute values of the elements, respectively. For simplicity, both of the norms will be denoted by $\|\cdot\|$. Especially (consider (a)–(c)), we have

(A) $\|M\| \geq 0$ and $\|M\| = 0$ if and only if $M = O$,
(B) $\|M \cdot N\| \leq \|M\| \cdot \|N\|$,
(C) $\|M \cdot u\| \leq \|M\| \cdot \|u\|$

for all $M, N \in Mat(F, m)$, $u \in F^m$. Henceforth, we will use properties (A)–(C) without emphasising.

The absolute value on F and the norms on F^m, $Mat(F, m)$ induce the metrics. For the sake of convenience, we will denote each one of these metrics by ϱ. The ε-neighbourhood of α will be denoted by $\mathcal{O}^{\varrho}_{\varepsilon}(\alpha)$ in all above mentioned spaces with metric ϱ. In addition, we assume that the valued field F (with $|\cdot|$) is separable. Hence, all considered spaces are separable. We remark that metric space (F, ϱ) does not need to be complete.

Let $\mathcal{X} \subset Mat(F, m)$ be a bounded group. We will study the homogeneous linear difference systems

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}, \quad \text{where} \quad \{A_k\} \subseteq \mathcal{X}. \tag{2}$$

Let $\mathcal{P}(\mathcal{X})$, $\mathcal{LP}(\mathcal{X})$, and $\mathcal{AP}(\mathcal{X})$ denotes the set of all systems (2) for which the sequence of matrices A_k is periodic, limit periodic, and almost periodic, respectively. Note that we will identify the sequence $\{A_k\}$ with the system (2) which is determined by $\{A_k\}$. In $\mathcal{AP}(\mathcal{X})$, we define the metric

$$\sigma(\{A_k\}, \{B_k\}) := \sup_{k \in \mathbb{Z}} \varrho(A_k, B_k), \quad \{A_k\}, \{B_k\} \in \mathcal{AP}(\mathcal{X}).$$

The symbol $\mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ will stand for the ε-neighbourhood of $\{A_k\}$ in $\mathcal{AP}(\mathcal{X})$.

3. Limit periodic, almost periodic, and asymptotically almost periodic sequences

Since we will consider various metric spaces, we recall the notion of limit periodicity, almost periodicity, and asymptotic almost periodicity for a general metric space (Y, τ). Note that, we mention only the properties of considered sequences which are necessary to prove our results.