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On quadrature of highly oscillatory integrals with logarithmic singularities[☆]

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ABSTRACT

In this paper a quadrature rule is discussed for highly oscillatory integrals with logarithmic singularities. At the same time, its error depends on the frequency ω and the computation of its moments are given. The new rule is implemented by interpolating f at Chebyshev nodes and singular point where the interpolation polynomial satisfies some conditions. Numerical experiments conform the efficiency for obtaining the approximations.

1. Introduction

In applied science, many models are the oscillatory integrals. This paper discusses the computation of highly oscillatory integrals

$$I[f] = \int_{-1}^{1} f(x)\log((x-\alpha)^2)e^{i\omega x}dx, \qquad (1.1)$$

where $\alpha \in [-1, 1]$, $f(x)$ is a sufficiently smooth real-valued function and $\omega \gg 1$. To compute (1.1) is difficult because of high oscillation and logarithmic singularities, which means that the previous methods may not be immediately used to the integral (see [1–19]).

Recently, a Filon–Clenshaw–Curtis approach

$$\mathcal{I}^{\alpha}_{k,N}[f] = \int_{-1}^{1} v(x)\log((x-\alpha)^2)e^{i\omega x}dx \approx I[f] \qquad (1.2)$$

is presented to evaluate (1.1) [20]. The polynomial $v(x)$ of degree N which interpolates f at Chebyshev nodes satisfies the conditions

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$$v\left(\cos\left(\frac{n\pi}{N}\right)\right) = f\left(\cos\left(\frac{n\pi}{N}\right)\right), \quad n = 0, 1, \ldots, N. \tag{1.3}$$

Domínguez's method (1.2) is effective to approximate (1.1), but the error only satisfies $O(\omega^{-1})$ for $\omega \gg 1$.

In this paper, our aim is to introduce a higher order method to obtain the approximation, where its moments can be evaluated by a simpler algorithm. Assume that the interpolation function $H(x)$ of $f(x)$ is the special Hermite interpolation polynomial of $f(x)$ at the Clewshaw–Curtis points $c_n = \cos(\frac{n\pi}{N})$ and singular point α, then the approximation of (1.1) can be denoted by

$$Q_s[f] = \int_{-1}^{1} H(x)\log((x-\alpha)^2)e^{i\omega x}dx. \tag{1.4}$$

In (1.4), $H(x)$ can be constructed according to the following three cases:

- If $\alpha = \pm 1$, then
$$H^{(k)}(-1) = f^{(k)}(-1), H(c_n) = f(c_n), H^{(k)}(1) = f^{(k)}(1) \tag{1.5}$$
where $n = 1, 2, \ldots, N-1$ and $k = 0, 1, \ldots, s$.
- If $-1 < \alpha < 1$ and $\alpha = \cos\frac{l\pi}{N}$ for $l \in [1, N-1]$, then
$$H^{(k)}(-1) = f^{(k)}(-1), H(c_n) = f(c_n), H^{(k)}(\alpha) = f^{(k)}(\alpha), H^{(k)}(1) = f^{(k)}(1) \tag{1.6}$$
where $n = 1, 2, \ldots, l-1, l+1, \ldots, N-1$ and $k = 0, 1, \ldots, s$.
- If $-1 < \alpha < 1$ and $\alpha \neq \cos\frac{l\pi}{N}$ for $\forall l \in [1, N-1]$, then
$$H^{(k)}(-1) = f^{(k)}(-1), H(c_n) = f(c_n), H^{(k)}(\alpha) = f^{(k)}(\alpha), H^{(k)}(1) = f^{(k)}(1) \tag{1.7}$$
where $n = 1, 2, \ldots, l_0-1, l_0+1, \ldots, N-1$, $k = 0, 1, \ldots, s$ and $c_{l_0} < \alpha < c_{l_0+1}$, $|c_{l_0} - \alpha| \le |c_{l_0+1} - \alpha|$.

The polynomials $H(x)$ defined by (1.4) can be expressed as

$$H(x) = \sum_{j=0}^{N+2s} a_j T_j(x), \quad as\ \alpha = \pm 1 \tag{1.8}$$

and

$$H(x) = \sum_{j=0}^{N+3s} a_j T_j(x), \quad as\ -1 < \alpha < 1, \tag{1.9}$$

where $T_j(x)$ is the Chebyshev polynomial of the first kind on $[-1, 1]$. Correspondingly, the general moments is that

$$M_j = \int_{-1}^{1} T_j(x)\log((x-\alpha)^2)e^{i\omega x}dx. \tag{1.10}$$

The outline of this paper is organized as follows: In Section 2 we give the algorithms to evaluate the moments (1.10) and the error analysis. Numerical examples are presented in Section 3 to demonstrate the results.

2. Computation of the moments and error analysis

For the Chebyshev polynomial $T_j(x)$ of the first kind or $U_j(x)$ of the second kind, there exists [23]

$$\begin{aligned}(1-x^2)P_j'(x) &= \frac{j}{2}(P_{j-1}(x) - P_{j+1}(x)),\\ 2xP_j(x) &= P_{j+1}(x) + P_{j-1}(x).\end{aligned} \tag{2.1}$$

Therefore, we get

$$-2xP_j(x) + (1-x^2)P_j'(x) = \left(\frac{j}{2} - 1\right)P_{j-1}(x) - \left(\frac{j}{2} + 1\right)P_{j+1}(x). \tag{2.2}$$

Let

$$K = \int_{-1}^{1}(1-x^2)T_j(x)\log((x-\alpha)^2)e^{i\omega x}dx. \tag{2.3}$$

Then

$$\begin{aligned} K &= \frac{1}{i\omega}\int_{-1}^{1}(1-x^2)T_j(x)\log((x-\alpha)^2)de^{i\omega x}\\ &= -\frac{1}{i\omega}\int_{-1}^{1}[(1-x^2)T_j(x)\log((x-\alpha)^2)]'e^{i\omega x}dx\\ &= -\frac{1}{i\omega}\int_{-1}^{1}[-2xT_j(x) + (1-x^2)T_j'(x)]\log((x-\alpha)^2)e^{i\omega x}dx - \frac{2}{i\omega}\int_{-1}^{1}\frac{(1-x^2)T_j(x)}{x-\alpha}e^{i\omega x}dx. \end{aligned} \tag{2.4}$$