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Central finite volume schemes on nonuniform grids and applications

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This paper is dedicated to the memory of professor Paul Arminjon (1941–2011).

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ABSTRACT

We propose a new one-dimensional unstaggered central scheme on nonuniform grids for the numerical solution of homogeneous hyperbolic systems of conservation laws with applications in two-phase flows and in hydrodynamics with and without gravitational effect. The numerical base scheme is a generalization of the original Lax–Friedrichs scheme and an extension of the Nessyahu and Tadmor central scheme to the case of nonuniform irregular grids. The main feature that characterizes the proposed scheme is its simplicity and versatility. In fact, the developed scheme evolves a piecewise linear numerical solution defined at the cell centers of a nonuniform grid, and avoids the resolution of the Riemann problems arising at the cell interfaces, thanks to a layer of staggered cells used intermediately. Spurious oscillations are avoided using a slopes limiting procedure. The developed scheme is then validated and used to solve classical problems arising in gas–solid two phase flow problems. The proposed scheme is then extended to the case of non-homogenous hyperbolic systems with a source term, in particular to the case of Euler equations with a gravitational source term. The obtained numerical results are in perfect agreement with corresponding ones appearing in the recent literature, thus confirming the efficiency and potential of the proposed method to handle both homogeneous and non-homogeneous hyperbolic systems.

1. Introduction

In this paper we develop a new unstaggered nonuniform central scheme (NUCS) for the numerical solution of general hyperbolic systems of conservation laws in one space dimension. Many problems arising in physics and engineering sciences can be formulated mathematically using hyperbolic systems of conservation laws [16,17,30] or, in the case of systems with a source term, hyperbolic systems of balance laws [29]. Such problems occur for example in aerodynamics, magnetohydrodynamics (MHD), hydrodynamics and many more. Central schemes are particularly attractive for solving hyperbolic systems as they avoid the resolution of the Riemann problems arising at the cell interfaces, thanks to a layer of staggered cells. Central schemes first appeared with the staggered version of Lax–Friedrichs's scheme, where a piecewise constant numerical solution was alternatingly evolved on two staggered grids. The resulting scheme is first-order accurate with a stability number of 0.5. In 1990 Nessyahu and Tadmor (NT) [19] presented a predictor-corrector type, second-order accurate scheme that is an extension of the Lax–Friedrichs scheme [14] which evolves a piecewise linear numerical solution on two staggered grids. The NT scheme uses a first-degree Taylor

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expansion in time to determine the numerical solution at the intermediate time; furthermore slopes limiting reduces spurious oscillations in the vicinity of discontinuities. Multidimensional extension of the Nessyahu and Tadmor scheme for hyperbolic systems on staggered Cartesian grids and on unstructured grids were later developed and successfully used to solve problems primarily arising in aerodynamics [1,2,12,13,31,35]. Central schemes for ideal and shallow water magnetohydrodynamics were later developed in [32]. Recently, unstaggered central schemes (UCS) for hyperbolic systems on Cartesian grids were developed in [33] where the numerical solution is evolved on a single grid and the resolutions of the Riemann problems at the cell interfaces is by-passed thanks to a layer of ghost staggered cells. With the UCS scheme, additional treatment of the obtained numerical solution can be easily performed without any synchronization problem since the numerical solution is computed on a unique grid at any time (for example in the case of ideal/shallow water MHD/SMHD and shallow water equation problems SWE [32–34]).

In this paper, we extend the unstaggered central schemes [31,33] to the case of one-dimensional Cartesian and nonuniform grids. More specifically, we extend the unstaggered central scheme to solve a system of partial differential equations related to two-phase fluid flow problems. Within this context, it is widely accepted that two-phase flow equations are of highly nonlinear, non-hyperbolic and of non-conservative type as an initial-boundary-value problem leading to unstable solutions [20,26]. In general, there are two different theoretical formulations in two-phase flow: The mixture theory and the averaging based approaches [7,8,11]. Based on such approaches, there exist a large number of improved two-phase flow models available in the literature that have been used successfully leading to a well-posed initial-boundary-value problem; see for example [15,23,28,36] and reference therein. An issue of concern, however, related to such models is that they inherently have a non-conservative form due to the interaction between phases [25,27]. A recent approach is based on the formulation of thermodynamically compatible systems [9]. In such an approach the two-phase flow character is formulated in terms of parameters of the mixture (see for example [22]). The resulting system of PDEs is fully hyperbolic and fully conservative as an initial-boundary value problem and the current paper focuses on this type of systems. However, as known to all, analytical solutions to two-phase flow equations are limited to special cases of such flows [4]. This is due to the non-linearity and complicated nature of the governing equations of interest. Thus, numerical approaches have been widely applied to investigate two-phase flow equations. For such reasons, in the remainder of this paper, we focus on the numerical resolutions of a specific two-phase flow equations. Recently, thermodynamically compatible systems theory has been introduced for problems related to gas–solid tow-phase flows [21,38] together with Godunov finite volume methods for the Riemann problem associated with the model equations. Here we develop another resolution to the model equations proposed in [38] based on the current numerical scheme developed in this paper.

We then extend the proposed central scheme to the case of systems of gas dynamics with a gravitational term, more precisely to the case of Euler equations with a gravitational source term. This system is a nonhomogeneous hyperbolic system with a source term describing the effect of gravity on the fluid momentum and energy. The numerical base scheme will be generalized to the case of non-homogenous system in general, and specifically to the case of Euler equations with gravity in the case of non-steady flows. Several methods were recently developed for the numerical solution of systems of Euler equations with gravity including, among others, the essentially non-oscillatory (ENO) finite volume and finite difference schemes [10], the weighted ENO finite volume and finite difference schemes [18], and the discontinuous Galerkin methods [3].

The rest of the paper is organized as follows: In Section 2 we develop the unstaggered nonuniform extension of the original central schemes, and then, in Section 3, we extend the scheme to the case of non-homogenous systems. In Section 4 we briefly discuss the derivation of the two-phase flow equations and the Euler equations with a gravitational source term. The numerical validation of the developed scheme and some classical two-phase gas–solid flow problems and Euler systems with gravity are presented in Section 5. Concluding remarks are given in Section 6.

2. Unstaggered central schemes on nonuniform grids

We consider the initial value problem:

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x, t = 0) = u_0(x). \end{cases} \tag{1}$$

$u(x, t) = (u_1, u_2, \ldots, u_p)$ is the unknown $p-$ components vector and $f(u)$ is the flux vector. System (1) is assumed to be hyperbolic, i.e., the Jacobian matrix $\partial f/\partial u$ has p real eigenvalues and p linearly independent eigenvectors [16,30]. We discretize the computational domain $[a, b]$ using n subintervals $C_i = [x_{i-1/2}, x_{i+1/2}]$ centered at the nodes x_i with different lengths Δx_i for $i = 1, \cdots, n$. The proposed numerical scheme evolves a piecewise linear numerical solution $L_i(x, t)$ defined at the centers x_i of the control cells C_i and we set $u_i^n = L_i(x_i, t^n)$ to be the cell centered value. The proposed scheme evolves the numerical solution on a unique grid with control cells C_i; however in order to avoid the resolution of the Riemann problems arising at the cell interfaces $x_{i\pm1/2}$, the numerical solution is first computed on the control dual cells $D_{ij} = [x_i, x_j]$ centered at x_{ij} with length Δx_{ij} and is then projected back onto the cells C_i. The geometry of the scheme is shown in Fig. 1. Without loss of generality we assume that the numerical solution u_i^n is known at time t^n at the center x_i of the cells C_i. The proposed scheme uses a piecewise linear reconstruction of the piecewise constant solution obtained at the previous time step:

$$u(x, t^n) \approx L_i(x, t^n) = u_i^n + (x - x_i)\,(u_i^n)', \ \ \forall x \in C_i, \tag{2}$$

where $(u_i^n)' \approx \frac{\partial}{\partial x} u(x, t^n)|_{x=x_i}$ approximates the slope to first-order accuracy. The solution of system (1) at time t^{n+1} is calculated as follows: We first integrate the conservation law in system (1) on the domain $R_{ij}^n = [x_i, x_j] \times [t^n, t^{n+1}]$ (shown in Fig. 2), and