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On input-to-state stability for stochastic coupled control systems on networks

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ABSTRACT

This paper investigates the input-to-state stability (ISS) properties for stochastic coupled control systems on networks (SCCSNs). With the help of Lyapunov method and graph theory, some inequality techniques and stochastic analysis skills, some sufficient criteria are obtained for $e^{\lambda t}$-weighted integral ISS in mean and almost sure exponential ISS of SCCSNs. These sufficient criteria are closely related to the topological property of the network anatomy. Finally, an example is given to illustrate the results.

1. Introduction

Coupled control systems on networks have come to play an important role in many fields, such as artificial intelligence, neural networks, engineering, etc. [1–4]. Stability analysis for coupled control systems on networks is one of the most important problems in control theory and engineering [5–8]. However, coupled control systems on networks in many applications are often perturbed by environmental noise in the real world. Some tiny noise often lead to the failure of stability for coupled control systems on networks. Hence, stability analysis for stochastic coupled control systems on networks (SCCSNs) have attracted the attention of many researchers in the past few years and lots of results have been reported (see [9–13]).

On the other hand, control systems are often perturbed by controls and errors on observations. Thus, it is desirable for a control system not only to be stable, but also to display Sontag's well-known 'input-to-state stability' (ISS) properties. The precise definition of ISS was introduced by Sontag in the later 1980s [14], and then he established some basic results [15,16]. Roughly speaking, so-called ISS means that the behavior of control systems should remain bounded when its inputs are bounded, and should tend to equilibrium when inputs tend to zero. In recent decades, ISS analysis of nonlinear control system has become one of the most active topic in nonlinear feedback analysis and design. Lots of important properties for ISS have been obtained by many researchers [17–22], and then these results have been applied in engineering, observer design, new small-gain theorems and control theory [23–26]. Thus, it is a necessary and meaningful task to discover the ISS properties for SCCSNs.

Among the methods that contributed to investigate stability of coupled control systems on networks, the Lyapunov method is the main one. Usually, linear matrix inequality and Lyapunov function are used to acquire stability criteria for coupled control systems on networks. However, how to construct an appropriate Lyapunov function for a specific coupled control systems on networks is still a difficult issue and this is the disadvantage of Lyapunov method. Recently, Li et al. investigated the global stability for a general coupled systems of differential equations on networks by graph theory [27,28]. By applying Kirchhoff's matrix tree theorem in graph theory, a systematic approach was given to constructed Lyapunov function for coupled systems on networks. This technology has been successfully employed in the global stability for many mathematic models on networks,

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such as, coupled oscillators model [29,30], multi-group model [31,32], and neural networks [33], etc. Moreover, this technology was also extended to many different systems, such as stochastic system [11,34,35], discrete-time system [36,37], and delay system [29,37,38]. But, to the best of the authors' knowledge, few scholars use this technology to investigate ISS properties for SCCSNs.

Motivated by the above discussions, in this paper, we aim to investigate ISS properties for SCCSNs. Some sufficient criteria ensuring the $e^{\lambda t}$-weighted integral ISS in mean and almost sure exponential ISS for SCCSNs are established by applying Lyapunov method and graph theory, some inequality techniques, and stochastic analysis skills. Finally, an example is provided to illustrate our results.

The main contribution of this paper is as follows.

1. We apply graph theory and Lyapunov method to investigate ISS properties for SCCSNs.
2. By employing Kirchhoff's matrix tree theorem in graph theory, some sufficient criteria are established which are closely related to the topological property of the network anatomy.

The rest of this paper is organized as follows. Some notations and preliminary results are given in next section. In Section 3, based on Lyapunov method and graph theory, the $e^{\lambda t}$-weighted integral ISS in mean and almost sure exponential ISS for SCCSNs are investigated. In Section 4, a numerical test is used to illustrate the results obtained in this paper. Finally, some conclusion remarks are drown in Section 5.

2. Notations and preliminary results

Throughout this paper, R denotes the set of all real numbers. Let $R_+ = [0, +\infty)$ and $Z^+ = \{1, 2, \ldots\}$. Set R^n and $R^{n\times m}$ denote the n-dimensional real space and $n \times m$-dimensional real matrix space, respectively. The transpose of vectors and matrices are denoted by superscript "T". For vector $x = (x_1, \ldots, x_n)^T \in R^n$, $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$ denotes the Euclidean norm. Let $\mathbb{L} = \{1, 2, \ldots, l\}$, $M = \sum_{i=1}^l m_i$ $(m_i \in Z^+)$, and $N = \sum_{i=1}^l n_i$ $(n_i \in Z^+)$. Let $C^{2,1}(R^d \times R_+; R)$ denote the family of all real-valued functions $V(x, t)$ defined on $R^d \times R_+$ such that they are continuously twice differentiable in x and once in t. A function α: $R_+ \to R_+$ is of class κ, if α is continuous, strictly increasing and $\alpha(0) = 0$. If α is also unbounded, then it is of class κ_∞.

Let $w(t)$ be a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration satisfying the usual conditions and $\mathbb{P}$ being a probability measure. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$.

Then, we introduce some basic concepts on graph theory [39,40]. A directed graph or digraph $\mathcal{G} = (H, E)$ contains a set $H = \{1, 2, \ldots, l\}$ of vertices and a set E of arcs (i, j) leading from initial vertex i to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc (j, i) is assigned a positive weight a_{ij}. In our convention, $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$. The weight $W(\mathcal{H})$ of a subgraph $\mathcal{H}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \ldots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, m-1\}$. If $i_m = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles, directed or undirected. A tree $\mathcal{T}$ is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. Given a weighted digraph $\mathcal{G}$ with l vertices, define the weight matrix $A = (a_{ij})_{l\times l}$ whose entry a_{ij} equals the weight of arc (j, i) if it exists, and 0 otherwise. Denote the directed graph with weight matrix A as $(\mathcal{G}, A)$. A digraph $\mathcal{G}$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_\mathcal{Q}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_\mathcal{Q}$ with $-\mathcal{C}_\mathcal{Q}$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1l} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{l1} & -a_{l2} & \cdots & \sum_{k\neq l} a_{lk} \end{pmatrix}.$$

The following result is standard in graph theory, and customarily called Kirchhoff's matrix tree theorem [39].

Lemma 1. *Assume that* $l \geq 2$. Let c_i *denote the cofactor of the ith diagonal element of L. Then*

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} W(\mathcal{T}), \quad i \in \mathbb{L},$$

where $\mathbb{T}_i$ *is the set of all spanning trees* $\mathcal{T}$ *of* $(\mathcal{G}, A)$ *that are rooted at vertex i, and* $W(\mathcal{T})$ *is the weight of* $\mathcal{T}$.

Other notations will be explained where they first appear. In the end of this section, we introduce an useful lemma as follows.

Lemma 2. [27]. *Assume that* $l \geq 2$. *Let* c_i *denote the cofactor of the i-th diagonal element of L. Then the following identity holds:*

$$\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_\mathcal{Q})} F_{rs}(x_r, x_s).$$