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Approximation of eigenvalues of Dirac systems with eigenparameter in all boundary conditions by sinc-Gaussian method

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ABSTRACT

In the present paper we apply a sinc-Gaussian technique to compute approximate values of the eigenvalues of Dirac systems and Dirac systems with eigenvalue parameter in one or two boundary conditions. The error of this method decays exponentially in terms of the number of involved samples. Therefore the accuracy of the new technique is higher than the classical sinc-method. Numerical worked examples with tables and illustrative figures are given at the end of the paper showing that this method gives us better results in comparison with the classical sinc-method in Annaby and Tharwat (2007, 2012) [5,6].

1. Introduction

Sampling theory is one of the most important mathematical tools used in communication engineering since it enables engineers to reconstruct signals from some of their sampled data. A fundamental result in information theory is the Whittaker–Kotelnikov–Shannon (WKS) sampling theorem [12,22,28]. It states that any $f \in \mathcal{B}_\sigma^2$, where $\mathcal{B}_\sigma^2$ is the space of all entire functions of exponential type $\sigma > 0$ which lie in $L^2(\mathbb{R})$ when restricted to $\mathbb{R}$, can be reconstructed from its sampled values $\{f(n\pi/\sigma) : n \in \mathbb{Z}\}$ by the formula

$$f(\lambda) = \sum_{n\in\mathbb{Z}} f(n\pi/\sigma)\,\mathrm{sinc}\,(\sigma\lambda - n\pi), \quad \lambda \in \mathbb{C}, \tag{1.1}$$

where

$$\mathrm{sinc}\,(\lambda) := \begin{cases} \dfrac{\sin(\lambda)}{\lambda}, & \lambda \neq 0, \\ 1, & \lambda = 0. \end{cases} \tag{1.2}$$

Series (1.1) converges absolutely and uniformly on compact subsets of $\mathbb{C}$, and uniform on $\mathbb{R}$, cf. [10]. Expansion (1.1) is used in several approximation problems which are known as sinc-methods, see e.g. [14,17,23,24]. In particular the sinc-method is used

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to approximate eigenvalues of boundary value problems, see for examples [3,5–8,26,27]. The sinc-method has a slow rate of decay at infinity, which is as slow as $O(|\lambda^{-1}|)$. There are several attempts to improve the rate of decay. One of the interesting ways is to multiply the sinc-function in (1.1) by a kernel function, see e.g. [9,11,25]. Let $h \in (0, \pi/\sigma]$ and $\gamma \in (0, \pi - h\sigma)$. Assume that $\Phi \in \mathcal{B}^2_\gamma$ such that $\Phi(0) = 1$, then for $f \in \mathcal{B}^2_\sigma$ we have the expansion, [21]

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f(nh)\operatorname{sinc}(h^{-1}\pi\lambda - n\pi)\Phi(h^{-1}\lambda - n). \tag{1.3}$$

The speed of convergence of the series in (1.3) is determined by the decay of $|\Phi(\lambda)|$. But the decay of an entire function of exponential type cannot be as fast as $e^{-c|x|}$ as $|x| \longrightarrow \infty$, for some positive c, [21]. In [18], Qian has introduced the following regularized sampling formula. For $h \in (0, \pi/\sigma]$, $N \in \mathbb{N}$ and $r > 0$, Qian defined the operator, [18]

$$(G_{h,N}f)(x) = \sum_{n \in Z_N(x)} f(nh)\, S_n(h^{-1}\pi x)\, G\left(\frac{x - nh}{\sqrt{2}\,rh}\right), \quad x \in \mathbb{R}, \tag{1.4}$$

where $G(t) := \exp(-t^2)$, which is called the Gaussian function, $S_n(h^{-1}\pi x) := \operatorname{sinc}(h^{-1}\pi x - n\pi)$, $Z_N(x) := \{n \in \mathbb{Z} : |[h^{-1}x] - n| \le N\}$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$, see also [19,20]. Qian also derived the following error bound. If $f \in \mathcal{B}^2_\sigma$, $h \in (0, \pi/\sigma]$ and $a := \min\{r(\pi - h\sigma), (N-2)/r\} \ge 1$, then [18,19]

$$|f(x) - (G_{h,N}f)(x)| \le \frac{2\sqrt{\sigma\pi}\,\|f\|_2}{\pi^2 a^2}\left(\sqrt{2\pi}\,a + e^{3/2r^2}\right)e^{-a^2/2}, \quad x \in \mathbb{R}. \tag{1.5}$$

In [21] Schmeisser and Stenger extended the operator (1.4) to the complex domain $\mathbb{C}$. For $\sigma > 0$, $h \in (0, \pi/\sigma]$ and $\omega := (\pi - h\sigma)/2$, they defined the operator, [21]

$$(\mathcal{G}_{h,N}f)(\lambda) := \sum_{n \in \mathbb{Z}_N(\lambda)} f(nh)\, S_n\left(\frac{\pi\lambda}{h}\right) G\left(\frac{\sqrt{\omega}(\lambda - nh)}{\sqrt{N}h}\right), \tag{1.6}$$

where $\mathbb{Z}_N(\lambda) := \left\{n \in \mathbb{Z} : |[h^{-1}\Re\lambda + 1/2] - n| \le N\right\}$ and $N \in \mathbb{N}$. Note that the summation limits in (1.6) depend on the real part of λ. Schmeisser and Stenger, [21], proved that if f is an entire function such that

$$|f(\xi + i\eta)| \le \phi(|\xi|)e^{\sigma|\eta|}, \quad \xi, \eta \in \mathbb{R}, \tag{1.7}$$

where ϕ is a non-decreasing, non-negative function on $[0, \infty)$ and $\sigma \ge 0$, then for $h \in (0, \pi/\sigma)$, $\omega := (\pi - h\sigma)/2$, $N \in \mathbb{N}$, $|\Im\lambda| < N$, we have

$$|f(\lambda) - (\mathcal{G}_{h,N}f)(\lambda)| \le 2\left|\sin(h^{-1}\pi\lambda)\right|\phi(|\Re\lambda| + h(N+1))\frac{e^{-\omega N}}{\sqrt{\pi\omega N}}\beta_N(h^{-1}\Im\lambda), \quad \lambda \in \mathbb{C}, \tag{1.8}$$

where

$$\beta_N(t) := \cosh(2\omega t) + \frac{2e^{\omega t^2/N}}{\sqrt{\pi\omega N}[1 - (t/N)^2]} + \frac{1}{2}\left[\frac{e^{2\omega t}}{e^{2\pi(N-t)} - 1} + \frac{e^{-2\omega t}}{e^{2\pi(N+t)} - 1}\right]. \tag{1.9}$$

The amplitude error arises when the exact values $f(nh)$ of (1.6) are replaced by the approximations $\widetilde{f}(nh)$. We assume that $\widetilde{f}(nh)$ are close to $f(nh)$, i.e. there is $\varepsilon > 0$, sufficiently small such that

$$\sup_{n \in \mathbb{Z}_n(\lambda)} \left|f(nh) - \widetilde{f}(nh)\right| < \varepsilon. \tag{1.10}$$

Let $h \in (0, \pi/\sigma)$, $\omega := (\pi - h\sigma)/2$ and $N \in \mathbb{N}$ be fixed numbers. The authors in [1] proved that if (1.10) is held, then for $|\Im\lambda| < N$, we have

$$\left|(\mathcal{G}_{h,N}f)(\lambda) - (\mathcal{G}_{h,N}\widetilde{f})(\lambda)\right| \le A_{\varepsilon,N}(\Im\lambda), \tag{1.11}$$

where

$$A_{\varepsilon,N}(\Im\lambda) = 2\,\varepsilon\; e^{-\omega/4N}\left(1 + \sqrt{N/\omega\pi}\right)\exp\left((\omega + \pi)h^{-1}|\Im\lambda|\right). \tag{1.12}$$

Consider the Dirac system which consists of the system of differential equations

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in [0, 1] \tag{1.13}$$

and the boundary conditions

$$\alpha_1 u_1(0) - \alpha_2 u_2(0) = -\lambda(\alpha_1' u_1(0) - \alpha_2' u_2(0)), \tag{1.14}$$

$$\beta_1 u_1(1) - \beta_2 u_2(1) = -\lambda(\beta_1' u_1(1) - \beta_2' u_2(1)), \tag{1.15}$$

where $r_1(\cdot), r_2(\cdot) \in L^1(0, 1)$ and $\alpha_i, \beta_i, \alpha_i', \beta_i' \in \mathbb{R}$, $i = 0, 1$, satisfying

$$((\alpha_1', \alpha_2') = (0, 0) \text{ or } \alpha_1'\alpha_2 - \alpha_1\alpha_2' > 0) \text{ and } ((\beta_1', \beta_2') = (0, 0) \text{ or } \beta_1\beta_2' - \beta_1'\beta_2 > 0). \tag{1.16}$$