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A numerical approach for solving generalized Abel-type nonlinear differential equations

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ABSTRACT

In this paper, a numerical power series algorithm which is based on the improved Taylor matrix method is introduced for the approximate solution of Abel-type differential equations and also, Riccati differential equations. The technique is defined and illustrated with some numerical examples. The obtained results reveal that the method is very effective, simple and valid high accuracy. The method can be easily extended to other nonlinear equations.

1. Introduction

Nonlinear differential equations are frequently used to model a wide class of problems in many scientific fields such as engineering, chemical reactions, mathematical physics, astrophysics, biology, ecology and economics. Most of these equations have no analytic solution and numerical techniques may be required to obtain approximate solutions. Thus, methods of solution for nonlinear differential equations are of great importance to engineers and scientists. In recent years, the attention has been focused on nonlinear problems; one of the most fundamental equations in the study of the nonlinear problems is nonlinear Abel-type first order differential equation which is defined in the general form [11–16],

$$A_1(x)y'(x) + A_2(x)y(x)y'(x) + A_3(x)y(x) + A_4(x)y^2(x) + A_5(x)y^3(x) = G(x), \quad a \le x \le b \tag{1}$$

where $A_j(x)$, $(j = 1, 2, 3, 4, 5)$ and $G(x)$ are continuous functions on $a \le x \le b$.

Eq. (1) plays an important role in many physical and mathematical problems, and technical applications. The properties of this equation have been intensively investigated in mathematical and physical literature [1–10]. Recently, Mak et al. [1,9,11] have presented a solution generating technique for Abel-type equations and shown that Lienard system describing nonlinear damped oscillations of a dynamical system can be reduced to a first order Abel differential equation. Also the second order nonlinear differential equation describing the dynamics connected with a bulk viscous cosmological fluid has been transformed into Abel-type differential equation [11]. These equations have been used to model the interior of static fluid spheres in general relativistic framework [4]. Mak and Harko [12] have been reduced the second order nonlinear evolution equation to Abel equation and have been obtained general solution in an exact form. Garcia et al. [13], by using the more stringent nonlinear second-order slow-roll approximation, have been converted the nonlinear second order equations of Stewart and Lyth to Abel equation. Alvarez et al. [14] have been obtained a new uniqueness criterion for the number of periodic orbits of Abel equations. Güler [15] has been proposed an algorithm based on the Taylor matrix method [17–24] for the Abel equation. On the other hand, some Taylor and

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Chebyshev (matrix and collocation) methods to solve linear and nonlinear differential, integral and integro-differential equations have been presented in many articles by Sezer and co-workers [17-27].

Our purpose in this study is to develop the above mentioned methods and apply to the generalized nonlinear Abel-type differential equation (1), which is a generalized case of the first order nonlinear Abel equations given in [11–15], with the initial condition

$$y(0) = \alpha \tag{2}$$

and to find the solution in terms of power series (Taylor polynomial form in origin)

$$y(x) = \sum_{n=0}^{N} y_n x^n, \quad y_n = \frac{y^{(n)}(0)}{n!}, \quad 0 \le x \le b \tag{3}$$

which is Taylor polynomial of degree N at $x = 0$, where $y_n, n = 0, 1, \ldots, N$ are unknown Taylor coefficients to be determined.

2. Fundamental relations

In this section we convert the expressions defined in (1)–(3) to the matrix forms by the following procedure: Firstly, the function $y(x)$ defined by (3) can be written in the matrix form

$$y(x) = \mathbf{X}(x)\mathbf{Y}, \tag{4}$$

where

$$\mathbf{X}(x) = \left[1\ x\ x^2 \ldots x^N\right], \quad \mathbf{Y} = [y_0\ y_1\ \ldots\ y_N]^T.$$

On the other hand, it is clearly seen that the relation between the matrix X(x) and its derivative $\mathbf{X}'(x)$ is

$$\mathbf{X}'(x) = \mathbf{X}(x)\mathbf{B}, \tag{5}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

From the matrix equations (4) and (5), it follows that

$$y'(x) = \mathbf{X}'(x)\mathbf{Y} = \mathbf{X}(x)\mathbf{B}\mathbf{Y}. \tag{6}$$

By using the production of two series, the matrix form of expression $y^2(x)$ is obtained as

$$y^2(x) = \tilde{\mathbf{X}}(x)\tilde{\mathbf{Y}}, \tag{7}$$

where

$$\tilde{\mathbf{X}}(x) = \left[\tilde{\mathbf{X}}_0(x)\tilde{\mathbf{X}}_1(x)\ \ldots\ \tilde{\mathbf{X}}_N(x)\right]_{1\times(N+1)^2},$$

$$\tilde{\mathbf{X}}_i(x) = \left[x^i x^0\ \ x^i x^1\ \ \cdots\ \ x^i x^N\right]\ or\ \tilde{\mathbf{X}}_i(x) = x^i\mathbf{X}(x),$$

and

$$\tilde{\mathbf{Y}} = \left[\tilde{\mathbf{Y}}_0\ \tilde{\mathbf{Y}}_1\ \ \ldots\ \ \tilde{\mathbf{Y}}_N\right]^T_{(N+1)^2\times 1}, \quad \tilde{\mathbf{Y}}_\mathbf{i} = [y_i y_0\ \ y_i y_1\ \ \cdots\ \ y_i y_N]^T, \quad i = 0, 1, \ldots, N.$$

In a similar way, we will have

$$y^3(x) = \approx{\mathbf{X}}(x)\approx{\mathbf{Y}} \tag{8}$$

where

$$\overset{\approx}{\mathbf{X}} = \left[\overset{\approx}{\mathbf{X}}_0(x)\ \ \overset{\approx}{\mathbf{X}}_1(x)\ \ \ldots\ \ \overset{\approx}{\mathbf{X}}_N(x)\right]_{1\times(N+1)^3}, \quad \underset{i}{\overset{\approx}{\mathbf{X}}}(x) = x^i\tilde{\mathbf{X}}(x).$$

$$\overset{\approx}{\mathbf{Y}} = \left[\overset{\approx}{\mathbf{Y}}_0\ \ \overset{\approx}{\mathbf{Y}}_1\ \ \cdots\ \ \overset{\approx}{\mathbf{Y}}_N\right]^T_{(N+1)^3\times 1}, \quad \underset{i}{\overset{\approx}{\mathbf{Y}}} = y_i\tilde{\mathbf{Y}},\ i = 0, 1, \ldots, N.$$

If we differentiate expression (7) with respect to$x$, we obtain

$$2y'(x)y(x) = \tilde{\mathbf{X}}'(x)\tilde{\mathbf{Y}} \tag{9}$$