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Numerical solution of non-linear Fokker–Planck equation using finite differences method and the cubic spline functions

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ABSTRACT

In this paper we proposed a finite difference scheme for solving the nonlinear Fokker–Planck equation. We apply a finite difference approximation for discretizing spatial derivatives. Then we use the cubic C^1-spline collocation method which is an A-stable method for the time integration of the resulting nonlinear system of ordinary differential equations. The proposed method has second-order accuracy in space and fourth-order accuracy in time variables. The numerical results confirm the validity of the method.

1. Introduction

Fokker–Planck equation arises in a number of different fields in natural science, including solid-state physics, chemical physics, quantum optics, theoretical biology, and circuit theory. A Fokker–Planck equation describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The Fokker–Planck equation was first used by Fokker and Planck (for instance, see [11]) to describe the Brownian motion of particles.

There is a more general form of Fokker–Planck equation. Nonlinear Fokker–Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing (see [1] and references therein).

In one variable case the nonlinear Fokker–Planck equation is written in the following form

$$\frac{\partial u(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t), \tag{1}$$

$$(x,t) \in [a,b] \times [0,T],$$

with initial condition

$$u(x,0) = \varphi(x),$$

and the boundary conditions

$$u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad t \geq 0.$$

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Where $A(x,t,u)$ is the drift vector and $B(x,t,u)$ is the diffusion tensor. We assume that ψ_1 and ψ_2 are smooth functions. Note that when $A(x,t,u)=A(x,t)$ and $B(x,t,u)=B(x,t)$ the nonlinear Fokker–Planck equation (1) reduces to the linear Fokker–Planck equation.

It is worth noting that some semi-analytic techniques are employed to solve the Fokker–Planck equation. For example this equation is investigated in [4] using the Adomian decomposition method. Also the variational iteration method is developed in [15] to solve this equation. For some other investigations on this model or some other similar models the interested readers can see references [5,6,9,10,16]. Authors of [18] developed a finite difference technique to solve the type of Fokker–Planck equations describing the stochastic dynamics of a particle in a storage ring.

In [17] a finite difference procedure is given for solving the Fokker–Planck equation in two dimensions. Lakestani and Dehghan in [7] proposed a numerical scheme for Fokker–Planck equation using the cubic B-spline scaling functions. For more applications of the model studied in this work the interested reader can see [2,3,8].

The organization of this paper is as follows. In Section 2, we present our method for solving nonlinear Fokker–Planck equations. Validation of this method is shown in Section 3 through some examples. A conclusion is drawn in Section 4.

2. Method of solution

In this section we will combine second-order central difference in space with cubic C^1-spline collocation method to obtain a high order method for solving the non-linear Fokker–Planck equation (1). At first we discretize partial differential equation (1) in space with central difference to obtain a system of ordinary differential equations with unknown function at each spatial grid point. Then we will apply the cubic C^1-spline collocation method for solving the resulting nonlinear system of ordinary differential equations (see [14]). Also this method can give a closed form approximation for the solution. For positive integers n and T, let $h=\frac{b-a}{n}$ denotes the step size of spatial derivatives and k denotes the step size of temporal derivative. So we define

$$x_i = ih, \quad i = 0,1,\ldots,n,$$
$$t_j = jk, \quad j = 0,1,\ldots.$$

We first rewrite the Eq. (1) as follows

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[u(x,t)A(x,t,u)] + \frac{\partial^2}{\partial x^2}[u(x,t)B(x,t,u)]. \tag{2}$$

We have the following relation

$$\frac{\partial A(x,t,u)}{\partial x} = \frac{\partial A}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial A}{\partial x}, \tag{3}$$

and

$$\frac{\partial^2 A(x,t,u)}{\partial x^2} = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial x\partial u}\frac{\partial u}{\partial x} + \frac{\partial A}{\partial u}\frac{\partial^2 u}{\partial x^2}. \tag{4}$$

Using Eqs. (3) and (4) in Eq. (2) we get:

$$\begin{aligned}\frac{\partial u}{\partial t} &= -u\left[\frac{\partial A}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial A}{\partial x}\right] - \frac{\partial u}{\partial x}A + u\left[\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial x\partial u}\frac{\partial u}{\partial x} + \frac{\partial B}{\partial u}\frac{\partial^2 u}{\partial x^2}\right] \\ &\quad + 2\frac{\partial u}{\partial x}\left[\frac{\partial B}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial B}{\partial x}\right] + \frac{\partial^2 u}{\partial x^2}B. \end{aligned}\tag{5}$$

Now we define

$$Q(x) = \frac{\partial u}{\partial t} + u\frac{\partial A}{\partial x} - u\frac{\partial^2 B}{\partial x^2}, \tag{6}$$

Then Eqs. (5) and (6) give

$$Q(x) = \frac{\partial u}{\partial x}\left[-u\frac{\partial A}{\partial u} - A + u\frac{\partial^2 B}{\partial x\partial u} + 2\frac{\partial B}{\partial x}\right] + \frac{\partial^2 u}{\partial x^2}\left[u\frac{\partial B}{\partial u} + B\right] + \left(\frac{\partial u}{\partial x}\right)^2\left(2\frac{\partial B}{\partial u}\right). \tag{7}$$

If we discretize the above equation with second-order central difference in space at each grid point, we obtain the following relation:

$$\begin{aligned} Q_i &= u_{i+1}\left[\frac{1}{2h}\left(-u\frac{\partial A}{\partial u} - A + u\frac{\partial^2 B}{\partial x\partial u} + 2\frac{\partial B}{\partial x}\right) + \frac{1}{h^2}\left(u\frac{\partial B}{\partial u} + B\right)\right]_i + u_i\left[\frac{-2}{h^2}\left(u\frac{\partial B}{\partial u} + B\right)\right]_i \\ &\quad + u_{i-1}\left[-\frac{1}{2h}\left(-u\frac{\partial A}{\partial u} + A + u\frac{\partial^2 B}{\partial x\partial u} + 2\frac{\partial B}{\partial x}\right) + \frac{1}{h^2}\left(u\frac{\partial B}{\partial u} + B\right)\right]_i \\ &\quad + \frac{u_{i+1}^2 - 2u_{i+1}u_{i-1} + u_{i-1}^2}{4h^2}\left[2\frac{\partial B}{\partial u}\right]_i. \end{aligned}\tag{8}$$