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# Efficient numerical techniques for Burgers' equation

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ABSTRACT

This paper presents new efficient numerical techniques for solving one dimensional quasi-linear Burgers' equation. Burgers' equation is used as a model problem in the study of turbulence, boundary layer behavior, shock waves, convection dominated diffusion phenomena, gas dynamics, acoustic attenuation in fog and continuum traffic simulation. Using a non-linear Cole–Hopf transformation the Burgers' equation is reduced to one-dimensional diffusion equation. The linearized diffusion equation is semi discretized by using method of lines (MOL) which leads to a system of ordinary differential equations in time. Resulting system of ordinary differential equations is solved by backward differentiation formulas (BDF) of order one, two and three and the analysis of numerical errors are presented. Numerical results for modest values of kinematic viscosity are compared with the exact solution to demonstrate the efficiency of proposed numerical methods.



## 1. Introduction

The nonlinear parabolic partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

is known as the one-dimensional Burgers' equation, where $\nu > 0$ is the kinematic viscosity parameter. It is the simplest nonlinear partial differential equation combining both nonlinear propagation and diffusive effects. When $\nu \neq 0$ Eq. (1.1) is known as viscid Burgers' equation and if $\nu = 0$, it is a non-linear hyperbolic partial differential equation also known as inviscid Burgers' equation [1].

Historically, Burgers' equation was first introduced by Bateman [7] in 1915 who derived the steady state solution for one dimensional Burgers' equation. This equation was proposed as a model of turbulent fluid motion [8] by the Dutch scientist Johannes Martinus Burgers and due to his remarkable contribution the equation is named after him. J.M Burgers' investigated various aspects of turbulence and used in [8,9] to model turbulence. This equation may be used to test various numerical algorithms, due to the availability of an analytical expression for its solution for different sets of boundary and initial conditions. A great deal of efforts have been expended in last few years to compute efficiently the numerical solution of the Burgers' equation for small and large values of the kinematic viscosity.

This equation was solved analytically for restricted values of initial conditions [14]. In 1972, Benton and Platzman [10] surveyed the exact solution to the initial value problem for the one-dimensional Burgers' equation. Ozin [23], proposed finite element method for the solution of Burgers' equation. In [25], local discontinuous Galerkin finite element method is used for solving Burgers' equation. Tangent method is used in [20], for analytic study of generalized Burgers' Huxley equation.

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We proposed to use discretization method for solving one-dimensional Burgers equation. It is one of the well known technique due to its simple concept and easy practice. Darvishi and Javidi [12] studied a numerical solution by pseudospectral method and Darvishi's preconditioning. Kadalbajoo and Awasthi [19] have developed a numerical method based on Crank–Nicolson scheme for Burgers' equation. Zhang and Wang [22] solved the Burgers' equation by a predictor-corrector compact finite difference scheme. Haq et al. [15] introduced meshless method of lines for the numerical solution of nonlinear Burgers' type equation. Gao [14] introduced lattice Boltzmann model for Burgers' equation through selecting equilibrium distribution function properly. In 2013, [18] a numerical scheme based on weighted average differential quadrature method is proposed to solve time dependent Burgers' equation with appropriate initial and boundary conditions. Some of the efficient numerical tools for the solution of Burgers' equation are Automatic differentiation method [3], Implicit finite difference scheme [4], group explicit methods [13], explicit methods [5,21] and finite elements methods [2,16].

In our context we will consider the following one-dimensional Burgers' equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad a \le x \le b, \quad 0 \le t \le T \tag{1.2a}$$

with initial condition

$$u(x,0) = u_0(x), \quad a \le x \le b, \tag{1.2b}$$

and boundary conditions

$$u(a,t) = f_1(t), \quad 0 \le t \le T, \tag{1.2c}$$

$$u(b,t) = f_2(t), \quad 0 \le t \le T. \tag{1.2d}$$

where $\nu > 0$, is the kinematic viscosity parameter and $u_0(x), f_1$ and $f_2$ are given functions of the variables which are sufficiently smooth.

In this paper, Cole–Hopf transformation is used to map the solution of quasi-linear Burgers' equation to the linear diffusion equation. The linear diffusion equation is semi-discretized by method of lines (MOL) into a system of first order ordinary differential equations. The resulting system is solved by different backward differentiation formulas.

The method of lines is a powerful numerical technique for the solution of time-dependent partial differential equations which was introduced by Schiesser in 1991 [24]. In this method semi discretization is performed along the spatial direction and the spatial derivatives are approximated by finite differences. Thus the partial differential equation is reduced to a system of ordinary differential equations which can be integrated in time. In this paper, we have used backward differentiation formulas of order one, two and three known as BDF-1, BDF-2 and BDF-3 to solve the system of ordinary differential equations. Numerical error analysis shows that the fully discretized scheme along with BDF-1 is first order accurate in time and second order accurate in space, while for BDF-2 errors are quadratic over both space and time. BDF-3 has got accuracy of order three in time and two in space.

Backward differentiation formulas belong to a family of linear multi-step implicit methods used for numerical integration of ordinary differential equations. Backward differentiation formula give an approximation to the derivative of a variable at a time '*t*' in terms of its function values, at time '*t*' and earlier times. To demonstrate the accuracy of these methods we have compared different schemes for modest values of kinematic viscosity.

## 2. Cole–Hopf transformation

J.D. Cole (1951) [11] and [17] gave a non-linear transformation to reduce Burgers' equation into linear diffusion equation. Cole–Hopf transformation is a powerful analytic tool for the Burgers' equation to yield exact solution. Recently, it has been recognized that Cole–Hopf transformation can also be used to find numerical solution. It appeared first in a technical report by Langerstrom, Cole and Trilling. Outlines of the Cole–Hopf transformation may be given by the following Theorem.

**Theorem 1.** *In the context with initial and boundary conditions of the Eq. (1.2) let $\phi(x,t)$ be any solution to linear diffusion equation*

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2},$$

*then the nonlinear transformation [Cole–Hopf]*

$$u = -2\nu\frac{\phi_x}{\phi}, \tag{2.3}$$

*is a solution to*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

Let

$$u = \frac{\partial \psi}{\partial x}, \qquad \psi = \psi(x,t)$$