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# New semilocal and local convergence analysis for the Secant method

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ABSTRACT

We present a new convergence analysis, for the Secant method in order to approximate a locally unique solution of a nonlinear equation in a Banach space. Our idea uses Lipschitz and center-Lipschitz instead of just Lipschitz conditions in the convergence analysis. The new convergence analysis leads to more precise error bounds and to a better information on the location of the solution than the corresponding ones in earlier studies such as [2,6,9,11,14,15,17,20,22–26]. Numerical examples validating the theoretical results are also provided in this study.



## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where $F$ is a Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A vast number of problems from Applied Science including engineering can be solved by means of finding the solution equations in a form like (1.1) using mathematical modeling [7,11,13,16,18,27]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations cannot be found in closed form. This is the main reason why the most commonly used solution methods are iterative. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since, all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence analysis one derives convergence criteria from the information around an initial point whereas in the local analysis one finds estimates of the radii of convergence balls from the information around a solution.

We consider the Secant method in the form

$$x_{n+1} = x_n - L_n^{-1}\, F(x_n) \quad (n \geq 0), \quad (x_{-1}, x_0 \in \mathcal{D}), \quad L_n = \delta F(x_n, x_{n-1}) \tag{1.2}$$

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where $\delta F(x,y) \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ $(x,y \in \mathcal{D})$ is the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ [16,20].

A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypotheses. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$. These are our objectives in this paper.

The Secant method, also known under the name of Regula Falsi or the method of Chords, is one of the most used iterative procedures for solving nonlinear equations. According to Ostrowski [21], this method is known from the time of early Italian algebraists. In the case of equations defined on the real line, the Secant method is better than Newton's method from the point of view of the efficiency index [7]. The Secant method was extended for the solution of nonlinear equations in Banach Spaces by Sergeev [26] and Schmidt [25].

The simplified Secant method

$$x_{n+1} = x_n - L_0^{-1}F(x_n) \qquad (n \geq 0), \quad (x_{-1}, x_0 \in D)$$

was first studied by S. Ulm [28]. The first semilocal convergence analysis was given by P. Laasonen [23]. His result was improved by Potra and Pták [24–26]. A semilocal convergence analysis for general Secant-type methods was given by Dennis [11]. Bosarge and Falb [10], Dennis [11], Potra [24–26], Argyros [[5–9]], Hernández et al. [14] and others [15,20,29], have also provided sufficient convergence conditions for the Secant method based on Lipschitz–type conditions on $\delta F$.

The use of Lipschitz and center-Lipschitz conditions is one way to enlarge the convergence domain of different methods. This technique consists of using both conditions together instead of using only the Lipschitz one. This allows to find a finer majorizing sequence. That is, a larger convergence domain. This technique has been used by Argyros in [8] in order to find weaker convergence criteria for Newton's method. Gutiérrez et al. in [13] give sufficient conditions for Newton's method using both Lipschitz and center-Lipschitz conditions, Magreñán in [19] for the damped Newton's method and Amat et al. in [3,4] or García–Olivo [12] for other methods. It turns out that our error bounds and the information on the location of the solution are more precise (under the same convergence condition (see (2.1))) than the old ones given in earlier studies such as [1,2,15,17,20,22–25,29,30].

The rest of the paper is organized as follows: The semilocal and local convergence analysis of the Secant method is presented in Section 2. Numerical examples are provided in Section 3.

## 2. Convergence analysis of the Secant method

In this section we shall first study the semilocal convergence analysis of the Secant method (1.2) for triplets $(F, x_0, x_{-1})$ belonging to the class $\mathcal{A} = A(k, k_0, k_1, k_2, b, c)$ defined as follows:

**Definition 2.1.** Let $k > 0$, $k_0 > 0$, $k_1 > 0$, $k_2 \geq 0$, $b \geq 0$ and $c \geq 0$ be parameters satisfying

$$k_1 b + 2\sqrt{k_1 c} \leq 1 \text{ and } \max\{k, k_2, k_0\} \leq k_1 \tag{2.1}$$

We say that a triplet $(F, x_0, x_{-1})$ belongs to the class $\mathcal{A}$, if:

$(\mathcal{A}_1)$ $F$ is a nonlinear operator defined on a convex subset $D$ of a Banach spaces $X$ with values in a Banach space $Y$.

$(\mathcal{A}_2)$ The points $x_0$ and $x_{-1}$ belong to the interior $D^\circ$ of $D$ and satisfy

$$\|x_0 - x_{-1}\| \leq b.$$

$(\mathcal{A}_3)$ $F$ is a Fréchet-differentiable on $D^\circ$ and there exists a mapping such that

$$\delta F : D^\circ \times D^\circ \to L(X,Y)$$

$$\delta F(x_0, x_{-1})^{-1}, F'(x_0)^{-1} \in L(Y,X),$$

$$\|\delta F(x_0, x_{-1})^{-1} F(x_0)\| \leq c, \tag{2.2}$$

$$\|F'(x_0)^{-1}(\delta F(x,y) - F'(x_0))\| \leq k_0\|x - x_0\| + k\|y - x_0\| \quad \text{for each } x, y \in D \tag{2.3}$$

and

$$\|F'(x_0)^{-1}(\delta F(x,y) - F'(z))\| \leq k_1\|x - z\| + k_2\|y - z\| \quad \text{for each } x, y, z \in D \tag{2.4}$$

$(\mathcal{A}_4)$ The set $D_C = \{x \in D;\ F \text{ is continuous at } x\}$ contains the ball $\bar{U}(x_0, t^* - b)$, where

$$t^* = \frac{1 + k_1 b - \sqrt{(1 + k_1 b)^2 - 4k_1(b + c)}}{2k_1} \tag{2.5}$$

It is convenient to associate with the class $\mathcal{A}$ the sequence $\{t_n\}$ defined by

$$t_{-1} = 0, \quad t_0 = b, \quad t_1 = b + c,$$
$$t_{n+1} = t_n - \frac{f(t_n)(t_n - t_{n-1})}{f(t_n) - f(t_{n-1})} \quad \text{for each } n = 1, 2, \ldots, \tag{2.6}$$

where

$$f(t) = k_1 t^2 - (1 + k_1 b)t + b + c.$$