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Global optimality conditions and optimization methods for constrained polynomial programming problems ☆

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ABSTRACT

The general constrained polynomial programming problem (*GPP*) is considered in this paper. Problem (*GPP*) has a broad range of applications and is proved to be NP-hard. Necessary global optimality conditions for problem (*GPP*) are established. Then, a new local optimization method for this problem is proposed by exploiting these necessary global optimality conditions. A global optimization method is proposed for this problem by combining this local optimization method together with an auxiliary function. Some numerical examples are also given to illustrate that these approaches are very efficient.

1. Introduction

The general constrained polynomial programming problem (*GPP*) is widespread in the mathematical modeling of real world systems for a very broad range of applications. Such applications include engineering design, signal processing, speech recognition, material science, investment science, quantum mechanics, allocation and location problems, quadratic assignment and numerical linear algebra [1,2]. Since polynomial functions are non-convex, the problem (*GPP*) is NP-hard, even when the objective function is quadratic and the feasible set is a simplex [3].

A classic approach for the problem (*GPP*) is convex relaxation methods [3–5]. Among various convex relaxation methods, semidefinite programming (SDP) and sum of squares (SOS) relaxations are very popular. Specifically, by representing each nonnegative polynomial as a sum of squares of some other polynomials, it is possible to relax each polynomial inequality as a convex linear matrix inequality(LMI) [6]. Theoretically, for Lasserre's SDP relaxation method, it was proved that when the feasible region of (*GPP*) is compact, its optimal value can be approximated within any accuracy by the sequence of SDP relaxations [7]. However, the practical solvability of SDP or SOS relaxation method depends on the size or the degree of the polynomial programming problem. Indeed, so far the most effective use of SDP relaxation has been for the quadratic optimization problems [6]. From a computational point of view, by SDP or SOS relaxation method, it is hard to achieve a close approximation to the optimal value of (*GPP*) without efficient methods for handling large scale semidefinite programs [7]. However, as the authors in [3] mentioned, so far there are few efficient numerical methods for solving large scale polynomial optimization problems. So, solving large scale SDPs still remains a computational challenge.

Recently, some researchers applied SDP relaxation methods to some special models. [8] provided approximation methods for complex polynomial optimization. In [8], the objective function takes three forms: multilinear, homogenous polynomial

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and a conjugate symmeric form. The constraint belongs to three sets: the mth roots of complex unity, the complex unity and the Euclidean sphere. [9] established some approximation solution methods to solve quadratically constrained multivariate bi-quadratic optimization. [6] presented a general semidefinite relaxation scheme for general n-variate quartic polynomial optimization under homogeneous quadratic constraints. [2] considered approximation algorithms for optimizing a generic multi-variate homogeneous polynomial function, subject to homogenous quadratic constraints.

Global optimality conditions are very important in global optimization field. References [10–13] focus on global optimality conditions for the problems with quadratic objective function subject to linear constraints or quadratic constraints. Based on the so-called Positivstellensatz (a polynomial analogue of the transposition theorem for linear systems), it is possible to formulate global necessary and sufficient conditions for problems (GPP) [14]. [15] proved in Theorem 4.2 a sufficient conditions for global optimality in (GPP), which is a special case of global necessary and sufficient conditions in [14]. [16] provided another necessary and sufficient global optimality conditions for (GPP). However all these conditions are complex and difficult to check in practice since the conditions involve solving a sequence of semidefinite programs. As it mentioned in [14], only under the idealized assumptions that all semidefinite programs can be solved exactly, it is possible for these conditions to be checked.

In this paper, we consider the following general constrained polynomial programming problem (*GPP*).

$$\begin{aligned} (GPP) \quad & \min f(x) \\ & s.t. \;\; g_t(x) \le 0, \quad t = 1, \dots, m \\ & \qquad x \in X, \end{aligned}$$

where $f: X \to R, g_t: X \to R, t = 1, \dots, m$, and X is a box with $x_i \in [u_i, v_i], i = 1, \dots, n$. $S = \{x \in X | g_t(x) \le 0, t = 1, \dots, m\}$ is the feasible set.

In this paper, we will discuss necessary global optimality conditions for problem (*GPP*). These conditions are obtained by studying Karush-Kuhn-Tucker (KKT) conditions and a necessary and sufficient condition for a point being a global minimizer for a constrained univariate polynomial programming problem. Then a new strongly local optimization method will be designed for problem (GPP) according to the necessary global optimality conditions. The new strongly local optimization method improves traditional local optimization methods which are based on KKT conditions. Finally, we will design a global optimization method to solve the problem (GPP) by combining the new strongly local optimization method and an auxiliary function. Numerical examples illustrate the efficiency of the optimization methods proposed in the paper.

The layout of the paper is as follows. Necessary global optimality conditions for the problem (*GPP*) are provided in Section 2. A new strongly local optimization method and a global optimization method for the problem (*GPP*) are designed in Section 3. Some numerical examples for the problem (*GPP*) are illustrated in Section 4. Conclusion is presented in Section 5.

2. Necessary global optimality conditions for problem (*GPP*)

In this section, we will provide necessary global optimality conditions for the problem (*GPP*). Actually, we construct a point set where the global minimizer lies in. We can obtain the global minimizer by comparing the function values of all points in the set.

Firstly, we consider the following univariate polynomial optimization.

$$\begin{aligned} (UPP) \; & \min \; p(x) \\ & s.t. \;\; q_t(x) \le 0, \quad t = 1, \dots, m \\ & \qquad x \in [u, v]. \end{aligned}$$

Let $\Omega = \{x \in [u, v] | q_t(x) \le 0, t = 1, \dots, m\}$.

The problem (*UPP*) is interesting not only because of the inherent simplicity of the problem strture and rich modeling capabilities, but also because this problem forms the backbone of multi-variate polynomial optimization [17].

For methods to solving the problem (*UPP*), please refer to [17,18] and the papers therein. [17] applies the global optimization algorithm (GOP) which proposed for solving constrained nonconvex problems involving quadratic and polynomial functions in the objective function and/or constraints in [19] to the special case of polynomial functions of one variable. It illustrates the effectiveness of the algorithm. [18] presents a significant enhancement of reformulation-linearization technique (RLT) and shows empirically that this approach yields very tight lower bounds.

Since, the feasible set Ω is a compact set and is not easy to work out, we will construct a new point set $\Omega^0 \subset \Omega$.

Let $\Omega^1 = \{u, v | q_t(u) \le 0, q_t(v) \le 0, t = 1, \dots, m\}$, $\Omega^2 = \{x | \nabla p(x) = 0, q_t(x) < 0, t = 1, \dots, m, x \in (u, v)\}$ and $\Omega_t^3 = \{x | q_t(x) = 0, q_j(x) \le 0, j \ne t, j = 1, \dots, m, x \in (u, v)\}$, $t = 1, \dots, m$. Let

$$\Omega^0 = \Omega^1 \bigcup \Omega^2 \bigcup_{t=1}^{m} \Omega_t^3. \tag{1}$$

Remark 1. Since $p(x)$ and $q_t(x), t = 1, \dots, m$, are univariate polynomials, we suppose that the degree of $p(x)$ is dp and the degrees of $q_t(x)$, are $dq_t, t = 1, \dots, m$, respectively. We use following methods to work out these point sets Ω^1, Ω^2 and Ω_t^3, $t = 1, \dots, m$:

1. u and v will be kept if $q_t(u) \le 0, q_t(v) \le 0, t = 1, \dots, m$. So, $|\Omega^1| \le 2$;
2. Calculate all stationary points of $p(x)$ in the interval (u, v) ($\{x \in (u, v) | \nabla p(x) = 0\}$) which will be kept if $q_t(x) < 0$, for all $t = 1, \dots, m$. So, $|\Omega^2| \le dp - 1$;