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An iterative method for solving general restricted linear equations

Shwetabh Srivastava*, Dharmendra K. Gupta

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India

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ABSTRACT

The Newton iterative method for computing outer inverses with prescribed range and null space is used in the non-stationary Richardson iterative method to develop an iterative method for solving general restricted linear equations. Starting with any suitably chosen initial iterate, our method generates a sequence of iterates converging to the solution. The necessary and sufficient conditions for the convergence along with the error bounds are established. The applications of the iterative method for solving some special linear equations are also discussed. A number of numerical examples are worked out. They include singular square, rectangular, randomly generated rank deficient matrices, full rank matrices and a set of singular matrices given in Matrix Computation Toolbox (mctoolbox) with the condition numbers ranging from order 10^{16} to 10^{50}. The mean CPU time (MCT) and the error bounds are the performance measures used. Our results when compared with the results obtained by Chen (1997) leads to substantial improvement in terms of both computational speed and accuracy.

1. Introduction

Let $\mathbb{C}_r^{m\times n}$, T and S denote the set of all $m \times n$ complex matrices of rank r, the subspace of $\mathbb{C}^n$ and the subspace of $\mathbb{C}^m$, respectively. Let $||A||_2$, A^t, A^*, $R(A)$ and $N(A)$ denote the matrix 2-norm, the transpose, the conjugate transpose, the range space and the null space of $A \in \mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, $\{1\}$-inverse of A is the matrix $A^{(1)} \in \mathbb{C}^{n\times m}$ satisfying $AA^{(1)}A = A$. Let $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer j such that $rank(A^{j+1}) = rank(A^j)$ is called the index of A and denoted by $\text{ind}(A)$. Let U and W be subspaces of $\mathbb{C}^n$ such that $U \oplus W = \mathbb{C}^n$. Then the sum $\oplus$ is called the direct sum of subspaces U and W. In addition, let $\rho(A)$ and $\lambda_{max}(A)$ denote spectral radius and the maximal eigenvalue of a square matrix A, respectively. In many problems of practical importance [2,3,10,13,18,19,21], one is concerned with an approximation of a solution of restricted linear equations

$$Ax = b, \tag{1.1}$$

where, $A \in C^{m\times n}$, $b \in \mathbb{C}^m$ and $x \in T$. By using the restricted generalized inverse of the matrix A relative to the subspace T, the expression by Ben-Israel and Greville [12] for the general solution of consistent linear system (1.1) is given as

$$x = P_T(AP_T)^{(1)}b + (I - P_T(AP_T)^{(1)}A)P_T w,$$

where, w is an arbitrary vector and P_T is the orthogonal projection of $\mathbb{C}^n$ on T. If A is a nonsingular square matrix, the solution of (1.1) is obtained by the classical Cramer rule. Recently, a number of methods [5,8,14,16] for the solutions of restricted linear

* Corresponding author. Tel.: +91 8016367291.
E-mail addresses: shwetabhiit@gmail.com (S. Srivastava), dkg@maths.iitkgp.ernet.in (D.K. Gupta).

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equations are studied in literature. Werner [15] considered Cramer rule for the unique solution of a special restricted linear Eq. (1.1) for $x \in K$, where, K is a complementary subspace of $N(A)$. Two different determinantal forms are derived, both reducing to the classical Cramer rule if A is nonsingular. The condensed Cramer rule for solving singular equations (1.1) for $x \in R(A^j)$, $b \in R(A^j)$ and $j = ind(A)$ is discussed in [6,17]. Sheng and Chen [4] presented the Cramer rule for (1.1) by using two maximum rank minor representations of the generalized inverse $A^{(2)}_{T,S}$. From [20], it is easy to see that the solution of (1.1) exists if and only if $b \in AT$ and it is unique if and only if $b \in AT$ and $T \cap N(A) = \{0\}$. Also, if $AT \oplus S = \mathbb{C}^m$, then $A^{(2)}_{T,S}b$ is a solution or the unique solution of (1.1) assuming it is consistent. A characterization of this special solution is established in [7] despite system (1.1) is solvable or not. Using a determinantal representation of the generalized inverse $A^{(2)}_{T,S}$, a formula for the unique solution of (1.1) for $x \in T$, $b \in AT$ and $dim(AT) = dim(T)$ is discussed in [9].

The iterative methods along with their convergence analysis are also developed for solving (1.1). It is well known that the iterative scheme of the form $x_{i+1} = Bx_i + c$, $i = 0, 1, \ldots$ where, B is an nth order complex matrix commonly called the iteration matrix is the most prevalent approach for solving (1.1). By using subproper splitting $A = M - N$, Wei et al. [7] developed iterative methods for approximating a solution of (1.1). Necessary and sufficient conditions on the splitting are given such that starting with any initial approximation x_0, the generated sequence converges to a solution for $b \in AT$. Further, if $b \in AT$ and $Z \in \mathbb{C}^{n\times m}$ satisfies $R(Z) \subset T$, then the sequence $\{x_k\}$ generated by the iterative method of Chen [1] for solving (1.1) is given for $k = 0, 1, 2, \ldots$ by

$$x_{k+1} = x_k + \beta Z(b - Ax_k), \tag{1.2}$$

where, β is a non zero real scalar. It converges to some solution of (1.1) starting with any initial approximation $x_0 \in T$ if and only if

$$\rho(P_{AT} - \beta AZ) < 1. \tag{1.3}$$

It should be noted that the iterative method (1.2) for solving (1.1) is linearly convergent. This is derived from the following general stationary method given by Richardson [24,25]

$$x_{k+1} = x_k + \omega(b - Ax_k),$$

where $\omega = \beta Z$ in [1].

In this paper, the non-stationary Richardson iterative method

$$x_{k+1} = x_k + \omega_k(b - Ax_k),$$

for solving (1.1) is used to generalize the method (1.2) by taking the sequence ω_k generated by the Newton iterative method [12] for computing outer inverses with prescribed range and null space. Starting with any initial approximation $x_0 \in T$ suitably chosen, the method generates a sequence converging to the solution. Convergence theorems giving necessary and sufficient conditions for the unique solution along with the estimation of error bounds are provided. The generalized method is then used for solving some special linear equations. It is also applied to obtain the Drazin inverse solution of (1.1). Numerical examples including singular square, rectangular, randomly generated rank deficient matrices, full rank matrices and a set of singular matrices given in Matrix Computation Toolbox (mctoolbox) [26] are worked out to demonstrate the efficacy of the method. The mean CPU time (MCT) and the error bounds are the performance measure used. Our results when compared with the results obtained by [1] leads to substantial improvement in terms of both computational speed and accuracy.

The paper is organized as follows. Section 1 is the introduction. In Section 2, some definitions and concepts used are introduced. The new iterative method for some solution of the general restricted linear equations is described in Section 3. A convergence theorem along with its error bounds and the application of the iterative method for finding the solutions of some special linear equations are also given. A number of numerical examples including singular square, rectangular, randomly generated rank deficient matrices, full rank matrices and a set of singular matrices obtained from the Matrix Computation Toolbox (mctoolbox) [26] are worked out in Section 4. Finally, conclusions are included in Section 5.

2. Definitions and concepts

In this section, we shall describe some definitions and concepts used in this paper.

Definition 2.1. Let $A \in \mathbb{C}^{m\times n}$ and let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively then a unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying $XAX = X$, $R(X) = T$ and $N(X) = S$, is called a {2}-inverse or outer inverse of A with prescribed range space T and null space S and is denoted by $A^{(2)}_{T,S}$.

Definition 2.2. Let $A \in \mathbb{C}^{m\times n}$, the Moore–Penrose inverse of A denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four equations

$$(i)\quad AXA = A, \quad (ii)\quad XAX = X, \quad (iii)\quad (AX)^* = AX, \quad (iv)\quad (XA)^* = XA.$$

Definition 2.3. For any $A \in \mathbb{C}^{n\times n}$, there is a unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$A^jXA = A^j, \qquad XAX = X, \quad AX = XA,$$

then X is called the Drazin inverse of matrix A and it is denoted by A^d, j is equal to the ind(A). If ind(A) = 1, then X is called the group inverse of A and is denoted by A_g.