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Numerical solutions of Volterra integral equation with weakly singular kernel using SCW method

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ABSTRACT

Numerical methods for weakly singular Volterra integral equations are rarely considered in the literature. The solutions of such equations may exhibit a singular behaviour in the neighbourhood of the initial point of the interval of integration and bring some difficulties in numerical computation. In this paper, we present a numerical solution of weakly singular Volterra integral equations including the Abels equations by the second Chebyshev wavelet (SCW). We give the SCW operational matrix of fractional integration, and combine with the block pulse functions (BPFs) to derive the procedure of solving this kind integral equations. The proposed method is illustrated with numerical examples. The results reveal that the method is accurate and easy to implement.

1. Introduction

In this paper, we propose a new method to solve the Volterra integral equation with weakly singular kernel

$$f(x)-\lambda\int_0^x \frac{k(x,t)F(f(t))}{(x-t)^{1-\alpha}}dt = g(x),\quad 0\le x\le 1. \tag{1}$$

The function $F(\cdot)$, $g(x)$ and kernel $k(x,t)$ are known, while real numbers $\alpha>0$ and λ define a type of equations. In the case when $0<\alpha<1$, this equation is also offered as the weakly singular Volterra integral equation (19), and above equation is a linear Volterra integral equation at $\alpha=1,\ F(f(t))=f(t)$. Particularly, if $k(x,t)=1,\ 0<\alpha=1-\beta<1$, we get the Abel integral equation:

$$f(x)-\lambda\int_0^x \frac{f(t)}{(x-t)^{\beta}}dt = g(x),\quad 0<\beta<1. \tag{2}$$

Weakly singular Volterra integral equations have many applications in various areas, such as mathematical physics, electrochemistry, semi-conductors, scattering theory, heat conduction, fluid flow and population dynamics [1,2]. There have been several numerical methods for the singular Volterra integral equations. For instance, product integration methods based on Newton–Cotes rules [3], Hermite-type collocation method [5], spline collocation and iterated collocation methods [4], extrapolation algorithm [6]. In recent years, researchers have turned their attention to solving weakly singular Volterra integral equations. However, only a few methods for this difficult topic are presented, such as Nyström interpolant method [7], graded mesh method [9], optimal homotopy asymptotic method [8], Tau method [11], Laplace transform and Taylor series [10].

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In recent years, wavelet theory has been widely applied in different fields of science and engineering. Wavelets permit the accurate representation of a variety of functions and operators, and establish a connection with fast numerical algorithms [12]. Orthogonal functions and polynomial series have been received considerable attention in dealing with various problems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problems. So the main objective of the present paper is solving the weakly singular Volterra integral equations based on operational matrix method.

The outline of this article is as follows. In Section 2 the basic formulation of SCW and some properties of SCW are given. In Section 3, we introduce some necessary definitions and mathematical preliminaries of fractional calculus and give the SCW operational matrix of fractional integration. In Section 4, we summarise the process of solving weakly singular Volterra integral equations based on the SCW operational matrix method. In Section 5, we provide several examples to show the efficiency and simplicity of the method. Concluding remarks are given in the last section.

2. The SCW and operational matrix of the fractional integration

2.1. The construction of SCW

The SCW which defined on the interval $[0,\ 1)$ have the following form [13,14]:

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{\mathrm{U}}_{\mathrm{m}}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $n = 1, \ldots, 2^{k-1}$ and k is any positive integer, and $\tilde{\mathrm{U}}_{\mathrm{m}}(t) = \sqrt{\frac{2}{\pi}}\mathrm{U}_m(t)$, here the coefficient $\sqrt{2/\pi}$ is used for orthonormality; $\mathrm{U}_m(t)$ is the second Chebyshev polynomials of degree m which respect to the weight function $\omega(t) = \sqrt{1-t^2}$. They are defined on the interval $[-1,\ 1]$ by the recurrence:

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \ldots$$

The weight function $\tilde{\omega}(t) = \omega(2t-1)$ has to be dilated and translated as $\omega_n(t) = \omega(2^k t - 2n + 1)$.

The SCW forms an orthonormal basis for $L^2[0, 1)$, i.e.

$$(\psi_{mn}(t), \psi_{m'n'}(t)) = \begin{cases} 1, & (m,n) = (m',n') \\ 0, & (m,n) \ne (m',n') \end{cases}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2_{\omega_n}[0,\ 1]$. The SCW has compact support $[(n-1)/2^{k-1}, n/2^{k-1}]$, $n = 1, \ldots, 2^{k-1}$.

2.2. Function approximation

A function $f(t)$ defined over [0, 1], may be expressed in terms of the SCW as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(t),$$

where the coefficients c_{nm} are given by

$$c_{nm} = (f(t), \psi_{nm}(t))_{\omega_n} = \int_0^1 \omega_n(t)\psi_{nm}(t)f(t)\,\mathrm{d}t.$$

We can approximate the function $f(t)$ by the truncated series

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^{\mathrm{T}}\Psi(t), \tag{4}$$

where the coefficient vector C and SCW function vector $\Psi(t)$ are given by

$$C = [c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}]^{\mathrm{T}}, \tag{5}$$

$$\Psi(t) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1(M-1)}, \psi_{20}, \ldots, \psi_{2(M-1)}, \ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1}(M-1)}]^{\mathrm{T}}. \tag{6}$$

Taking the collocation points as following

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M. \tag{7}$$

We define the SCW matrix $\Phi_{m'\times m'}$ as

$$\Phi_{m'\times m'} = \left[\Psi\left(\frac{1}{2m'}\right), \Psi\left(\frac{3}{2m'}\right), \ldots, \Psi\left(\frac{2m'-1}{2m'}\right)\right],$$