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Gap function and global error bounds for generalized mixed quasi variational inequalities

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ABSTRACT

In this paper, we obtain some gap functions for generalized mixed quasi variational inequality problems in terms of regularized gap function and *D*-gap function. Further, by using these gap functions we obtain global error bounds for the solution of generalized mixed quasi variational inequality problems in Hilbert spaces. The results obtained in this paper generalize and improve some corresponding known results in literatures.

1. Introduction

In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities related to set-valued operators, non convex optimization and non monotone operators. A useful and important generalization of variational inequalities is a mixed variational inequality containing the nonlinear term. For the applications of the mixed variational inequalities, see for example [12–14,16,19–22,24–26] and the references therein. Due to the presence of the nonlinear term, one cannot develop the projection-type algorithms for solving the mixed quasi-variational inequalities, which motivated authors to develop another technique. This technique is related to the resolvent of the maximal monotone operator. The main idea of this technique was introduced by Brezis [4]. Further by using the concept of the resolvent operator technique, many authors introduced and studied the various resolvent equations to develop the sensitivity analysis for mixed variational inequalities.

One of the classical approach in the analysis of variational inequality problem is to transform it into an equivalent optimization problem via the notion of gap function, see for example [1–3,7,9–12,14,17,21–25,27] and the references therein. This enables us to develop descent-like algorithms to solve variational inequality problem. Besides these, gap functions also turned out to be very useful in designing new globally convergent algorithms, in analyzing the rate of convergence of some iterative methods and in obtaining the error bounds. Gap functions have turned out to be very useful in deriving the error bounds, which provide a measure of the distance between solution set and an arbitrary point. Recently, many error bounds for various kinds of variational inequalities have been established, see for example [1,3,7,9–12,17,21–25,27] and the references therein.

Throughout this paper, let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $C(H)$ be a family of nonempty compact subsets of H. Let $S, T, : H \to C(H)$ be the set-valued operators and $g : H \to H$ be a

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single-valued operator. Let $\phi(\cdot,\cdot): H \times H \to \mathbb{R} \cup \{+\infty\}$ be a continuous bifunction with respect to both arguments. Let $F: H \times H \to \mathbb{R}$ be a bifunction satisfying $F(x,x) = 0$, for all $x \in H$. For given nonlinear operator $N(\cdot,\cdot): H \times H \to H$, we consider the following *generalized mixed quasi variational inequality problem*, denoted by GMQVIP, which consists in finding $x \in H$, $u \in S(x)$, $v \in T(x)$ such that

$$F(g(x), g(y)) + \langle N(u,v), g(y) - g(x)\rangle + \phi(g(x), g(y)) - \phi(g(x), g(x)) \geq 0, \quad \forall y \in H. \tag{1.1}$$

The quasi variational inequality problems are definitely most notable one among the several variants of variational inequality problems. An important reason for this is that a number of problems involving the non convex, and nonsmooth operators arising in optimization, mechanics and structural engineering theory can be studied via the generalized mixed quasi variational inequalities, see for example [2,3,6,12,22] and the references therein.

If $g \equiv I$, the identity operator and $F \equiv 0$, then GMQVIP(1.1) is equivalent to *generalized mixed set-valued variational inequality problem*, denoted by GMSVVIP, which consists in finding $x \in H$, $u \in S(x)$, $v \in T(x)$ such that

$$\langle N(u,v), y - x\rangle + \phi(x,y) - \phi(x,x) \geq 0, \quad \forall y \in H, \tag{1.2}$$

a problem studied by Noor [19–21] using the auxiliary principle techniques.

If $\phi(x,y) = \phi(y)$, $S \equiv 0$ and $T: H \to C(H)$ are set-valued operator, $N(u,v) = T(x)$, then problem GMSVVIP(1.2) collapses to *set-valued mixed variational inequality problem*, denoted by SVMVIP, which consists in finding $x \in H$ such that

$$\exists u \in T(x): \langle u, y - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in H, \tag{1.3}$$

which was considered by Tang [25]. They introduced two regularized gap functions for above SVMVIP(1.3) and studied there differentiable properties.

If T is single valued, then problem SVMVIP(1.3) reduces to *mixed variational inequality problem*, denoted by MVIP, which consists in finding $x \in H$ such that,

$$\langle T(x), y - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in H, \tag{1.4}$$

which was studied by Solodov [24]. In this paper, he introduced three gap functions for MVIP(1.4) and by using these he obtained error bounds.

If the function $\phi(\cdot)$ is an indicator function of a closed set K in H, then problem MVIP(1.4) reduces to *set-valued variational inequality problem*, denoted by SVVIP, which consists in finding $x \in K$ such that,

$$\exists u \in T(x): \langle u, y - x\rangle \geq 0, \quad \forall y \in K, \tag{1.5}$$

studied by Li [17]. They obtained some existence results for global error bounds for gap function under strong monotonicity. Later, Aussel [1], defined gap functions and by using it they obtained finiteness and error bounds properties for above set-valued variational inequalities.

If T is single valued and $K: H \to C(H)$ be a set-valued mapping, such that $K(x)$ is a closed convex set in H, for each $x \in H$, then above problem SVVIP(1.5) is equivalent to *quasi variational inequality problem*, denoted by QVIP, which consists in finding $x \in K(x)$ such that

$$\langle T(x), y - x\rangle \geq 0, \quad \forall y \in K(x), \tag{1.6}$$

which was studied by Gupta [11] and Noor [21]. They derived local and global error bounds for above quasi variational inequality problems in terms of the regularized gap function and the D-gap function.

Inspired and motivated by the recent research work above, we introduce gap functions and error bounds for generalized mixed quasi variational inequality problems. Since this class is the most general and includes the previously studied some classes of variational inequalities as special cases, therefore our results cover and extend the previously known results under weaker conditions. The results presented in this paper generalize and improve the work presented in [12,21,24,25].

This paper is organized as follows: In Section 2, we give some basic definitions and results which will be used in this paper. Further we define normal residual vector $R(x,\theta)$ to derive the global error bounds for the solution of GMQVIP(1.1). In Section 3, we introduce a regularized gap function for GMQVIP(1.1) and derived error bounds without using Lipschitz continuity assumption. In Section 4, we introduce D-gap function and derive error bounds for the solution of the GMQVIP(1.1) under some weaker conditions.

2. Preliminaries and basic facts

In order to establish resolvent equations for the GMQVIP(1.1), we needed the following definitions and results.

Definition 2.1. Let $F: H \times H \to \mathbb{R}$ and $\phi: H \times H \to \mathbb{R}$ be two bifunctions. Then

(a) F is said to be monotone if, $F(x,y) + F(y,x) \leq 0, \quad \forall x, y \in H$;
(b) ϕ is said to be skew-symmetric if, $\phi(x,x) - \phi(x,y) - \phi(y,x) + \phi(y,y) \geq 0, \quad \forall x, y \in H.$

Remark 2.1. Clearly if the skew-symmetric bifunction $\phi(\cdot,\cdot)$ is bilinear, then $\phi(x,x) \geq 0$, $\forall x \in H$. In fact,

$$\phi(x,x) - \phi(x,y) - \phi(y,x) + \phi(y,y) = \phi(x-y, x-y) \geq 0, \quad \forall x, y \in H.$$