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Simultaneous approximation with generalized Durrmeyer operators ☆

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ABSTRACT

The aim of this paper is to obtain some convergence properties of generalized sequences of Ibragimov–Gadjiev–Durrmeyer operators which are a wide class of linear positive operators including many well known linear positive operators. Firstly, the Voronovskaya type theorem in simultaneous approximation is given. Then we present an upper estimate of norm convergence of the derivatives of the operators in quantitative mean in terms of the modulus of continuity. We show several of sequences that can be derived from them by means of a suitable transformation. Some special cases of new operators are presented as examples.

1. Introduction

In the theory of approximation by linear positive operators, the researchers have studied many operators within the domain of approximated functions and the space of approximated functions. Amongst others, the most important of these operators are Bernstein, Szasz, Baskakov, etc. Of course, it is useful to define a new sequence of linear positive operators in such a way that obtaining results can be received for a wide class of operators and validate each of them simultaneously. For this purpose, in 1970, new linear positive operators were defined by Ibragimov and Gadjiev [10]. They first showed that defined operators are approximation processes and they studied their shape preserving properties. We can find a large number of papers devoted to the study of properties of the convergence of these operators. For more details we refer the readers to [4] and [5,6,8,9,11,12]. However, till now the approximation properties of the operators defined by Ibragimov–Gadjiev have been studied within the space of continuous functions and their subspaces. In order to furnish an approximation process for spaces of integrable functions on unbounded intervals, Aral and Acar [3] introduced the Durrmeyer modification of Gadjiev and Ibragimov operators, called Ibragimov–Gadjiev–Durrmeyer operators, and studied some convergence results. We emphasize that our considered operators are different from the Baskakov–Durrmeyer type operators given by Heilmann [16] and the Srivastava–Gupta operators given in [21]. We just restricted ourselves in this paper with the Durrmeyer operators acting on unbounded intervals because of assumptions of main results. However, in general, the operators also include the Durrmeyer operators acting on compact intervals. For more details for Durrmeyer type operators, we refer the readers to [13].

It is important to determine whether the operators reproduce the properties of the function that we are trying to approximate. For this purpose, showing they approximate not only the function but also all its derivatives we shall obtain, they present good simultaneous approximation properties. We also established a Lorentz-type lemma for these operators and focused on the Voronovskaya type theorem for the derivative of any order of approximated function. Then the norm convergence of any

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derivative of Ibragimov–Gadjiev–Durrmeyer to derivative of approximated function in quantitative mean is given. Let us recall the above-mentioned operators.

Definition 1. Let $(\varphi_n(t))_{n\in\mathbb{N}}$ and $(\psi_n(t))_{n\in\mathbb{N}}$ be sequences of functions in $C(\mathbb{R}^+)$, which is the space of continuous functions on $\mathbb{R}^+$, such that $\varphi_n(0)=0$, $\psi_n(t)>0$, for all t and $\lim_{n\to\infty} 1/n^2\psi_n(0)=0$. Also let $(\alpha_n)_{n\in\mathbb{N}}$ denote a sequence of positive numbers satisfying the conditions

$$\lim_{n\to\infty}\frac{\alpha_n}{n}=1 \quad \text{and} \quad \lim_{n\to\infty}\alpha_n\psi_n(0)=l_1, \quad l_1\geq 0.$$

The Ibragimov–Gadjiev–Durrmeyer operators are defined by

$$\begin{aligned} M_n(f;x) = {} & (n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\ & \times\int_0^\infty f(y)K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy, \end{aligned} \tag{1.1}$$

$K_n^{(\nu)}(x,0,\alpha_n\psi_n(0)) := \frac{\partial^\nu}{\partial u^\nu}K_n(x,0,u)\Big|_{u=\alpha_n\psi_n(t),t=0}$ is a sequence of functions of three variables x,t,u, such that for each $x,t\in\mathbb{R}^+=[0,\ \infty)$ and for each $n\in\mathbb{N}$, K_n is the entire analytic function with respect to variable u, satisfying following conditions:

(1) Every function of this sequence is an entire function with respect to u for fixed $x,t\in\mathbb{R}^+$ and $K_n(x,0,0)=1$ for $x\in\mathbb{R}^+$ and $n\in\mathbb{N}$.
(2) $[(-1)^\nu K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))]\geq 0$ for $\nu=0,1,\ldots$ and $x\in\mathbb{R}^+$.
(3) $K_n^{(\nu)}(x,0,u_1)=-nx[K_n^{(\nu-1)}(x,0,u_1)]$ for all $x\in\mathbb{R}^+$ and $n\in\mathbb{N}$, $\nu=0,1,\ldots,m$ is a number such that $m+n=0$ or a natural number. (This notation means that the derivative with respect to u is taken ν times, then one set $u=u_1$ and $t=0$)
(4) $K_n(0,0,u)=1$ for any $u\in\mathbb{R}$ and

$$\lim_{x\to\infty}x^pK_n^{(\nu)}(x,0,u_1)=0,$$

for any $p\in\mathbb{N}$ and fixed $u=u_1$.
(5) For any fixed t and u the function $K_n(x,t,u)$ is continuously differentiable with respect to variable $x\in\mathbb{R}^+$ and satisfying the equality

$$\frac{d}{dx}K_n(x,0,u_1)=-nu_1K_{m+n}(x,0,u_1)$$

for fixed $u=u_1$.
We assume that the function $K_n(x,t,u)$ in addition to the conditions (1)–(5) satisfy also the condition:
(6) $\frac{n+\nu m}{1+u_1mx}K_n^{(\nu)}(x,0,u_1)=nK_{n+m}^{(\nu)}(x,0,u_1)$ for all $x\in\mathbb{R}^+$, $n\in\mathbb{N}$, $\nu=0,1,\ldots$ and fixed $u=u_1$.

Since $K_n(x,t,u)$ is entire function with respect to variable u by the assumption (1), we can write the Taylor expansion for $K_n(x,t,u)$ at any point $u_1\in\mathbb{R}$ as

$$K_n(x,t,u)=\sum_{\nu=0}^{\infty}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1}\frac{(u-u_1)^\nu}{\nu!}$$

and replacing $u=\varphi_n(t)$, $u_1=\alpha_n\psi_n(t)$ and $t=0$, where (α_n) is the sequence defined in (1.1),

$$K_n(x,0,0)=\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\varphi_n(0))^\nu}{\nu!}$$

is obtained by the condition $\varphi_n(0)=0$. Taking into account that $K_n(x,0,0)=1$ by the condition (1), we have

$$\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\varphi_n(0))^\nu}{\nu!}=1. \tag{1.2}$$

We remark that the operators M_n are linear and positive. It is well known that the operators M_n preserve the degree of polynomials and convexity.

Throughout the paper, we consider that these functions belong to the class of all Lebesgue measurable functions f on $\mathbb{R}^+$, that is,

$$\mathcal{H}\equiv\left\{f:\int_0^\infty\frac{|f(t)|}{(1+ct)^{n/c}}dt<\infty,\ \text{for } c>0 \text{ and some } n\in\mathbb{N}\right\}$$

with the norm $\|\cdot\|_{C_\alpha}$ given by $\|f\|_{C_\alpha}=\sup_{t\in\mathbb{R}^+}\frac{|f(t)|}{t^\alpha}$.

Considering this space several researchers studied simultaneous approximation properties of some other operators. In this direction we refer to [1,2,7,14,17] and references therein.