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Existence of solutions of functional integral equations of convolution type using a new construction of a measure of noncompactness on $L^p(\mathbb{R}_+)$

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ABSTRACT

Let $L^p(\mathbb{R}_+)$ denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ with the standard norm

$$\|x\|_p = \left(\int_0^\infty |x(t)|^p dt\right)^{\frac{1}{p}}.$$

First, we define a new measure of noncompactness on the spaces $L^p(\mathbb{R}_+)$ $(1 \le p < \infty)$. In addition, we study the existence of entire solutions for a class of nonlinear functional integral equations of convolution type using Darbo's fixed point theorem, which is associated with the new measure of noncompactness. We provide some examples to demonstrate that our results are applicable whereas the previous results are not.

1. Introduction

Nonlinear functional integral equations of convolution type play important roles in applied problems, especially numerous branches of mathematical physics such as neutron transportation, radiation, and gas kinetic theory (see [23] and the references therein). Equations of this type have been considered in many previous studies [5,22], which showed that these equation have solutions in some function spaces. In addition, Banás and Knap [8] discussed the solvability of the equations considered in the space of Lebesgue integrable functions using the technique of measures of weak noncompactness and the fixed point theorem due to Emmanuel [19]. This approach gives more general results under less restrictive assumptions compared with those in [5,22], but the weak continuity conditions for an operator are not readily satisfied in general. However, in 1955, Darbo presented a fixed point theorem [13] that uses the measure of noncompactness technique, which then became a tool for investigating the existence and behavior of solutions of many classes of integral equations such as Volterra, Fredholm, and Uryson type integral equations on $C[a,b]$, $BC(\mathbb{R}_+)$ and $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ (see [1–4,6,9–11,15–17,20]). In this study, we aim to overcome this difficulty by using Darbo's fixed point theorem and a new measure of noncompactness on the spaces $L^P(\mathbb{R}_+)$ $(1 \le p < \infty)$. We also investigate the existence of entire solutions of a class of nonlinear functional integral equations of convolution type on $L^P(\mathbb{R}_+)$ $(1 \le p < \infty)$, whereas previous studies only considered those on $L^1(\mathbb{R}_+)$ (see [8,14,18,20]).

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We recall the definition of the measure of noncompactness based on some terms of natural conditions. Let $\mathbb{R}_+ = [0, +\infty)$ and $(E, \|\cdot\|)$ be a Banach space. The symbols $\overline{X}$ and $ConvX$ denote the closure and closed convex hull of a subset X of E, respectively. Moreover, let us denote $\mathfrak{M}_E$ as the family of nonempty bounded subsets of E and $\mathfrak{N}_E$ as the subfamily comprising all relatively compact subsets of E.

Definition 1.1. ([7]). A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

1° The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subseteq \mathfrak{N}_E$.
2° $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(ConvX) = \mu(X)$.
5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0, 1]$.
6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$, for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

The following Darbo's fixed point theorem is needed in Section 3.

Theorem 1.1. ([13]). *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, and let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that a constant $k \in [0, 1)$ exists with the property*

$$\mu(FX) \le k\mu(X), \tag{1.1}$$

for any nonempty subset X of Ω. Then, F has a fixed point in the set Ω.

In the main part of our results (Section 2), we define new measures of noncompactness on the spaces $L^P(\mathbb{R}_+)$ and we study its property. Finally (Section 3), using the obtained results in Section 2, we investigate the problem of the existence of solutions for a class of nonlinear integral equations and we present some examples.

2. Main results

Let $L^p(\mathbb{R}_+)$ denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ with the standard norm

$$\|x\|_p = \left(\int_0^\infty |x(t)|^p dt\right)^{\frac{1}{p}}.$$

Before introducing the new measures of noncompactness on the spaces $L^p(\mathbb{R}_+)$, we need to characterize the compact subsets of $L^p(\mathbb{R}_+)$.

Theorem 2.1. ([12,21]). *Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \le p < \infty$. The closure of $\mathcal{F}$ in $L^p(\mathbb{R}^N)$ is compact if and only if*

$$\lim_{h\longrightarrow 0} \|\tau_h f - f\|_p = 0 \quad uniformly \ in \ f \in \mathcal{F}, \tag{2.1}$$

where $\tau_h f(x) = f(x+h)$ for all $x, h \in \mathbb{R}^N$. In addition, for $\varepsilon > 0$, there is a bounded and measurable subset $\Omega \subset \mathbb{R}^N$ such that

$$\|f\|_{L^p(\mathbb{R}^N \setminus \Omega)} < \varepsilon, \qquad \text{for all } f \in \mathcal{F}. \tag{2.2}$$

The following corollary is a special case of the theorem above, which we use in the sequel.

Corollary 2.2. *Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}_+)$ with $1 \le p < \infty$. The closure of $\mathcal{F}$ in $L^p(\mathbb{R}_+)$ is compact if and only if*

$$\lim_{h\longrightarrow 0} \left(\int_0^\infty |f(x) - f(x+h)|^p dx\right)^{\frac{1}{p}} = 0 \quad uniformly \ in \ f \in \mathcal{F}. \tag{2.3}$$

In addition, for $\varepsilon > 0$, there is a constant $T > 0$ such that

$$\left(\int_T^\infty |f(x)|^p dx\right)^{\frac{1}{p}} < \varepsilon, \quad \text{for all } f \in \mathcal{F}. \tag{2.4}$$

Now, we are ready to define a new measure of noncompactness on the spaces $L^p(\mathbb{R}_+)$.

Theorem 2.3. *Suppose that $1 \le p < \infty$ and X is a bounded* subset of *$L^p(\mathbb{R}_+)$. For $x \in X$ and $\varepsilon > 0$, let*

$$\omega(x, \varepsilon) = \sup\left\{\left(\int_0^\infty |x(t+h) - x(t)|^p dt\right)^{\frac{1}{p}} : |h| < \varepsilon\right\}$$
$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$$
$$\omega(X) = \lim_{\varepsilon\to 0} \omega(X, \varepsilon)$$