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Convergence analysis of compact difference schemes for diffusion equation with nonlocal boundary conditions ☆

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ABSTRACT

Compact difference schemes for solving the diffusion equation with nonlocal boundary conditions are considered in this paper. Fourth-order compact difference is used to approximate the second order spatial derivative, and the integrals in the boundary conditions are approximated by the composite Simpson quadrature formula. The backward Euler and Crank–Nicolson schemes are presented as the fully discrete schemes. Error estimates in the discrete h^1 and l^∞ norms are given by the energy method, showing both schemes are fourth-order accurate in space, and they have first-order and second-order accuracy in time, respectively. Numerical results are provided to confirm the theoretical results.

1. Introduction

In this paper, we study the compact difference schemes for the one dimensional, time-dependent linear heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < 1, \quad 0 < t \le T \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 1 \tag{2}$$

and the integral boundary conditions

$$\begin{aligned} u(0,t) &= \int_0^1 \alpha(x)u(x,t)\,dx + \varphi(t), \\ u(1,t) &= \int_0^1 \beta(x)u(x,t)\,dx + \psi(t), \quad 0 < t \le T. \end{aligned} \tag{3}$$

The above problem arises in the quasi-static theory of thermoelasticity [1–3], and the existence, uniqueness and some analytic properties of the solution of (1)–(3) have been studied (e.g., [1,2]) under the assumption

$$\int_0^1 |\alpha(x)|\,dx < 1 \quad \text{and} \quad \int_0^1 |\beta(x)|\,dx < 1. \tag{4}$$

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For the heat equation with an integral term in the boundary condition, many numerical methods have been proposed and studied in the recent years. For example, the θ-scheme which is second-order accurate in space, was discussed with some efficient algorithms for solving the resulting linear system of equations [4]. Pao [5] gave some monotone iterative schemes for a class of nonlinear reaction–diffusion equations with nonlocal boundary conditions and analyzed them using the discrete maximal and minimal solutions techniques. Ekolin [6] proved the convergence of the Crank–Nicolson scheme subject to the condition

$$\left(\int_0^1 \alpha^2(x)\,dx\right)^{\frac{1}{2}} + \left(\int_0^1 \beta^2(x)\,dx\right)^{\frac{1}{2}} < \frac{\sqrt{3}}{2}. \tag{5}$$

For $\frac{1}{2} \le \theta \le 1$, Liu [4] proved the convergence of the θ-scheme under the condition

$$\int_0^1 \alpha^2(x)\,dx + \int_0^1 \beta^2(x)\,dx < 2. \tag{6}$$

Dehghan [7–9] proposed some finite difference schemes, and reviewed on various approaches in the literature for the numerical solution of the one-dimensional heat equation subject to the specification of mass.

Besides the finite difference method, there are some other numerical methods on this subject, e.g., Abbasbandy and Shirzadi [10,11] present the meshless local Petrov– Galerkin method to treat the two-dimensional diffusion equations with non-classical boundary conditions. For problems with nonlocal boundary conditions involving integrals of the unknown solution over the spatial interval, a finite element scheme for the second-order elliptic problem was constructed recently [12,13], and orthogonal spline collocation method for the heat equation was presented in [14] with optimal order error estimates.

Compared with the difference schemes with second-order convergence in space, compact finite difference schemes have high-order accuracy and the desirable tridiagonal nature of the finite-difference equations, and these schemes have been used efficiently to solve various kinds of partial differential equations [15–21]. For non-local boundary value problems, high-order stable three-level schemes for hyperbolic equations were discussed in [22], and some two-level fourth-order explicit algorithms for diffusion and diffusion–reaction problems were derived [23–26], with the stability proved using Fourier method and the error estimates given by Lax equivalence theorem after the local truncation errors being analyzed.

To the author's knowledge, error estimate in the discrete energy norm for the compact Crank–Nicolson scheme was provided in [27] under a very restrictive condition

$$\left(\int_0^1 \alpha^2(x)\,dx\right)^{\frac{1}{2}} + \left(\int_0^1 \beta^2(x)\,dx\right)^{\frac{1}{2}} < \sqrt{0.432}. \tag{7}$$

Therefore, our purpose is to prove the convergence in the discrete h^1-norm and l^∞-norm under a weaker condition, for the two fourth-order compact schemes presented in this paper. To give the convergence analysis, besides (4), we assume that the functions $\alpha(x)$ and $\beta(x)$ satisfy

$$\int_0^1 \alpha^2(x)\,dx + \int_0^1 \beta^2(x)\,dx < 1. \tag{8}$$

This restriction is weaker than (7).

The paper is organized as follows. In Section 2, we replace the spatial derivative by the fourth-order compact difference, then approximate the time derivative by the backward difference and the central difference, respectively. That is, we focus on two special cases of the general θ-schemes. Since the boundary conditions include the integrals of the unknown variable over the entire spatial domain, we use the composite Simpson quadrature rule to give a discretization, thus we obtain two implicit compact difference schemes. Both schemes are fourth-order accurate in space, one scheme is the backward Euler scheme(which is first order accurate in time), another one is the Crank–Nicolson scheme(which has second order accuracy in time). In Section 3, error estimates are given using the energy method. Finally, some numerical examples are given in Section 4 to verify the theoretical conclusions. The conclusion is given in Section 5.

Throughout this paper, the symbols ϵ and C are generic positive constants, they are independent of the mesh sizes, and may take different values at different places.

2. Construction of the compact difference schemes

2.1. Partition of the domain

For the numerical solution of (1)–(3) we introduce a uniform grid of mesh points (x_j, t_n), with $x_j = jh, j = 0, 1, \ldots, 2N_x$ and $t_n = n\tau, n = 0, 1, \ldots, N$. Here N_x and N are positive integers, $h = 1/(2N_x)$ is the mesh-width in x, and $\tau = T/N$ is the time step. For any function $v(x,t)$, we let $v_j^n = v(x_j, t_n)$, e.g., the theoretical solution u at the mesh point (x_j, t_n) is denoted by u_j^n, and U_j^n stands for the solution of an approximating difference scheme at the same mesh point.