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Soliton solutions to a class of relativistic nonlinear Schrödinger equations ☆

Yongkuan Cheng*, Yangxin Yao

Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, PR China

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ABSTRACT

By using a change of variables, we get new equations, whose respective associated functionals are well defined in $H^1(\mathbb{R}^N)$ and satisfy the geometric hypotheses of the mountain pass theorem. Using this fact, we obtain a nontrivial solution.

1. Introduction

We study the existence of solutions for the following quasilinear Schrödinger equations

$$-\triangle u + V(x)u - \frac{\alpha}{2}[\triangle(1+u^2)^{\frac{\alpha}{2}}]\frac{u}{(1+u^2)^{\frac{2-\alpha}{2}}} = u^q + u^p, \quad x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $2T(\alpha)+2 < q+1 < p+1 < \alpha 2^* := 2\alpha N/(N-2)$, $\alpha \geq 1$, $T(\alpha)$ is defined in Lemma 2.1 listed below, and here the potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies

(V_0) $V(x) \geq V_0 > 0$, for all $x \in \mathbb{R}^N$.
(V_1) $\lim_{|x|\to\infty} V(x) = V(\infty)$ and $V(x) \leq V(\infty)$, for all $x \in \mathbb{R}^N$.

The solutions of (1) are related to the existence of solitary wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\triangle z + W(x)z - h(|z|^2)z - \triangle l(|z|^2)l'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{2}$$

where W is a given potential, l and h are real functions. Quasilinear equations such as (2) have been accepted as models of several physical phenomena corresponding to various types of l. The case of $l(s) = s^\alpha$ was used for the superfluid film equation in plasma physics [1]. Besides, (2) also appear in plasma physics and fluid mechanics [2], in dissipative quantum mechanics [3], in the theory of Heisenberg ferromagnetism and magnons [4,5]. See also [6,7] for more physical backgrounds. Eqs. (2) with $l(s) = s$, that is, $\alpha = 1$, have been studied extensively recently, see [8,9] and [17]. When $l(s) = (1+s)^{\frac{\alpha}{2}}$, (2) turn into our equations (1) with $h(s) = s^q + s^p$. Especially, if we let $\alpha = 1$, that is, $l(s) = (1+s)^{\frac{1}{2}}$, (2) models the self-channeling of a high-power ultrashort laser in matter [10]. In this case, few results are known. In [11], the authors proved global existence and uniqueness of small solutions in transverse space dimensions 2 and 3, and local existence without any smallness condition in transverse space dimension 1. In [18], the authors proved the existence of nontrivial solution. When $\alpha > 1$, although we do not know the physical background of Eqs. (2), in a mathematical sense, we give the proof of the existence of nontrivial solutions.

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* Corresponding author. Tel.: +86 20 87110448; fax: +86 20 87110448.
E-mail address: chengyk@scut.edu.cn (Y. Cheng).

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Here, our special interest is the existence of solutions of type $z(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that $z(t,x)$ satisfies (2) if and only if the function $u(x)$ satisfies (1), where $V(x) \doteq W(x) - E$ is the new potential. As described in [12], $z(t,x)$ was called the solitary wave solutions of (2) and the corresponding $u(x)$ was called the soliton solutions of (1).

For (1), the main difficulty is that the energy functional associated with (1) is not well defined in $H^1(\mathbb{R}^N)$. To overcome this difficulty, enlightened by [8,9], we give a new change of variables. Then we reduce the quasilinear problem (1) to a semilinear one, which we will prove has a nontrivial solutions.

Our main result is the following

Theorem 1.1. *Assume that* $\alpha \geq 1,\ 2T(\alpha) + 2 < q + 1 < p + 1 < \alpha 2^*$ *and* $(V_0) - (V_1)$ *hold, then (1) has a nontrivial solution.*

In this paper, C denotes positive (possibly different) constant, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm $|u|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{\frac{1}{p}},\ 1 \leq p < \infty$, $H^1(\mathbb{R}^N)$ denotes the Sobolev space with norm $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx)^{\frac{1}{2}}$.

2. The change of variables

We note that the solutions of (1) are the critical points of the following functional

$$\begin{aligned} I(u) = &\frac{1}{2}\int_{\mathbb{R}^N}\Big[1 + \frac{\alpha^2 u^2}{2(1+u^2)^{2-\alpha}}\Big]|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx \\ &- \frac{1}{q+1}\int_{\mathbb{R}^N} u^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}dx. \end{aligned} \tag{3}$$

Since the functional $I(u)$ may not be well defined in the usual Sobolev spaces $H^1(\mathbb{R}^N)$. We make a change of variables as

$$v = G(u) = \int_0^u g(t)\,dt,$$

where $g(t) = \sqrt{1 + \frac{\alpha^2 t^2}{2(1+t^2)^{2-\alpha}}}$. Since $g(t)$ is monotonous with $|t|$, the inverse function $G^{-1}(t)$ of $G(t)$ exists. Then after the change of variables, $I(u)$ can be written by

$$\begin{aligned} J(v) = &\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \\ &- \frac{1}{q+1}\int_{\mathbb{R}^N} |G^{-1}(v)|^{q+1}dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |G^{-1}(v)|^{p+1}dx. \end{aligned} \tag{4}$$

By Lemma 2.1 listed below, we have $\lim_{t\to 0} G^{-1}(t)/t = 1$ and $\lim_{t\to\infty}|G^{-1}(t)|^\alpha/t = \sqrt{2}(\alpha > 1)$ or $\sqrt{2/3}(\alpha = 1)$, so $J(v)$ is well defined in $H^1(\mathbb{R}^N)$ and $J(v) \in C^1$.

If u is a nontrivial solution of (1), then for all $\phi \in C_0^\infty(\mathbb{R}^N)$ it should satisfy

$$\int_{\mathbb{R}^N} [g^2(u)\nabla u \nabla\phi + g(u)g'(u)|\nabla u|^2\phi + V(x)u\phi - u^q\phi - u^p\phi]\,dx = 0. \tag{5}$$

We show that (5) is equivalent to

$$\begin{aligned} J'(v)\psi = &\int_{\mathbb{R}^N}\Big[\nabla v \nabla\psi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi - \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))}\psi \\ &- \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))}\psi\Big]dx = 0, \quad \forall\psi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{6}$$

Indeed, if we choose $\phi = \frac{1}{g(u)}\psi$ in (5), then we get (6). On the other hand, since $u = G^{-1}(v)$, if we let $\psi = g(u)\phi$ in (6), we get (5). Therefore, in order to find the nontrivial solutions of (1), it suffices to study the existence of the nontrivial solutions of the following equations

$$-\triangle v = -V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} + \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))} + \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{7}$$

Before we close this section, we give some properties of the change of variables.