Applied Mathematics and Computation 260 (2015) 370–384

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical analysis of bump solutions for neural field equations with periodic microstructure

Elena Malyutina, Arcady Ponosov, John Wyller*

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, P.O. Box 5003, NO-1432 Ås, Norway

A R T I C L E I N F O

Keywords:
Rate equations in neuroscience
Periodic
Homogenization

A B S T R A C T

We study numerically single bump solutions of a homogenized Amari equation with periodic microvariation. Two attempts are made to detect single bumps that depend on the microvariable. The first attempt which is based on a pinning function technique is applicable in the Heaviside limit of the firing rate function. In the second attempt, we develop a numerical scheme which combines the two-scale convergence theory and an iteration procedure for the corresponding heterogeneous Amari equation. The numerical simulations in both attempts indicate the nonexistence of single bump solutions that depend on the microvariable. Motivated by this result, we finally develop a fixed point iteration scheme for the construction of single bump solutions that are independent of the microvariable when the firing rate function is given by a sigmoidal firing rate function.

1. Introduction

The Amari equation [1]

$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{-\infty}^{\infty} \omega(x-y)f(u(y,t))\,dy \tag{1}$$

is one of the simplest nonlocal field models for the spatiotemporal variation of the neural activity in cortical networks. Here u denotes the average neural activity, ω the coupling strength (referred to as the connectivity function) and f the firing rate function. The actual networks are modelled as a continuous sheet of neurons where the typical spatial and temporal scales of the activity are assumed to be much larger than the corresponding neuronal scales.

The model (1) as well as its modifications and extensions have been used as starting points for the study of traveling waves and localized stationary solutions (so-called *bumps*) and the stability of these coherent structures. A common assumption made in such investigations is that the firing rate function is approximated with the unit step function (Heaviside function). This is convenient from the mathematical point of view as one in that case can find closed form analytical expressions for the traveling waves and the bump solutions. One then carries out the corresponding stability assessment either by a phase space reduction method involving the projection of the dynamics of the full system onto a finite dimensional space in the crossing coordinates between the bumps profiles and the threshold values (i.e. the so-called *Amari approach*) or by means of full stability analysis (i.e. the *Evans function approach*). Moreover, even though no rigorous mathematical justification is given, one tacitly assumes that the Heaviside limit of the firing rate function produces results which represent a sensible approximation to the results obtained

* Corresponding author. Tel.: +47 67231632.
E-mail addresses: elena.malyutina@nmbu.no, lenokimfi@gmail.com (E. Malyutina), arkadi@nmbu.no (A. Ponosov), john.wyller@nmbu.no (J. Wyller).

http://dx.doi.org/10.1016/j.amc.2015.03.058

in steep but continuous firing rate case. The special case consisting of a one-to-one correspondence between the admissible threshold values and the pulsewidth coordinates, leads to localized stationary solutions called *a single bump solution* (or *a 1-bump solution*) of the model (1). The problem of existence and uniqueness of these solutions has been studied together with their linear stability. See for example Coombes [2] and Bressloff [3] and the references therein.

The problem of deviation from the Heaviside limit of the firing rate function can be resolved in the case when the spectrum of the connectivity kernel is a rational function. In that case the Amari equation (1) can be converted to an ODE which possesses a Hamiltonian structure. The existence of bumps is then resolved by means of standard techniques for such systems i.e. detecting the homoclinic orbits of the system. See for example [4] and [5]. In the general case with no specific assumptions imposed on the connectivity kernel, existence of stationary solutions is proved using functional analytical techniques (fixed point theorems) as well as their dependence on the steepness parameter of the firing rate function is explored [6–9]. For the purpose of constructing the actual stationary solutions one has to rely on different numerical schemes. Coombes and Schmidt [10] has proposed an iteration scheme for construction of single bump solutions of the Amari equation (1) in the case of a sigmoidal firing rate function, without actually given any rigorous justification of the approach. In Oleynik et al. [11] two iteration schemes for constructing single bump solutions in the case of sigmoidal firing rate function are proposed. A rigorous justification of the convergence properties of these schemes is also given. The first scheme is a bumps width iteration method which generalizes the method proposed in Coombes Schmidt [10] while the second one is a fixed point iteration procedure based on Kishimoto Amari [6]. In [7] the spatial domain is assumed to be bounded and the firing rate functions sigmoidal. Here the fixed point structure of the stationary problem is exploited in the numerical construction procedure for the bumps solutions.

The modelling framework (1) and many of its extensions presuppose that the medium is homogeneous and isotropic. Hence one does not take into account the microscopic fine structure which obviously is present in the cortex. Modelling of these effects is therefore important and has hence been subject to much research effort in the neuroscience community. See for example Bressloff [3] and the references therein. One way of tackling the coupling of macro- and microstructure problem in neural field models is by using *homogenization techniques* [12,13]. When the medium possesses a periodic microstructure, the homogenization results in an averaging over some well identified microscale. In the neural field theory context the coupling between periodic micro level structure of the cortex and nonlocal mean field description has been investigated in the works [14–19]. It turns out that the detailed microstructure has an impact on pattern forming mechanisms as well as existence and stability of traveling fronts and pulses.

Standard homogenization techniques consist of different type of perturbation expansions (see for example Persson et al. [20] and the references therein). Modern homogenization theory based on multi-scale convergence theory represents an alternative and rigorous approach to this problem. It yields efficient and rigorous methods for studying the coupling between the microstructure and macroscopic levels. The multiscale technique was originally presented by Nguetseng [21]. A review of the method is given in Lukkassen et al. [22].

In [23–25] it is shown that the nonlocal neural field model

$$\frac{\partial}{\partial t}u_\varepsilon(x,t) = -u_\varepsilon(x,t) + \int_\Omega \omega\left(x'-x, \frac{x'-x}{\varepsilon}\right) f(u_\varepsilon(x',t))\,dx' \tag{2}$$

converges to

$$\frac{\partial}{\partial t}u(x,y,t) = -u(x,y,t) + \int_\Omega dx' \int_Y dy'\omega(x'-x, y'-y)f(u(x',y',t)) \tag{3}$$

in the two-scale sense when $\varepsilon \to 0$. Here $x \in \Omega \subseteq \mathbb{R}^N$, $t > 0$. The connectivity kernel ω by assumption is periodic in the second argument $y = \frac{x}{\varepsilon}$, i.e. $\omega_\varepsilon(x) = \omega(x, \frac{x}{\varepsilon})$. A key feature in the derivation of (3) from (2) is the exploitation of Visintin's theorem on two-scale convergence of convolution integrals [26]. This result enables us to get the correct limit of the convolution term in (2) as $\varepsilon \to 0$. Svanstedt et al. [24] construct the y-independent single bump solutions of the homogenized equation (3) by using the pinning function technique in the case of Heaviside step firing function. In the same paper stability theory for these bumps is developed. Just as in the translational invariant case (1) intervals for which the pinning function is increasing correspond to unstable bumps, while for the complementary regimes with a decreasing pinning function the corresponding bumps are stable.

This serves as a background for the present paper.

In this paper we first of all give numerical evidence for the nonexistence of y-dependent single bump solutions. We demonstrate this in two different ways: First of all, we use a pinning function technique which generalizes the method developed in Svanstedt et al. [24]. This method is applicable to the case when the firing rate function is given by means of the Heaviside step function. Secondly, we make use of a combination of the iteration method for single bumps developed in Oleynik et al. [11] and the two-scale convergence superposition method described in Visintin [26]. The latter method is designed for constructing single bump solutions of the homogenized Amari equation (3) in the case of the sigmoidal firing rate function. Then, as consequence of these two attempts to construct single bumps of (3) with a nontrivial variation in the local variable y, we embark on formulating a numerical scheme for iterative construction of y-independent single bumps of (3).

We organize the present paper as follows: In Section 2 we present the stationary versions of the models (2) and (3) and the assumptions imposed on the connectivity kernel and the firing rate function. Section 3 is devoted to the two attempts to construct for y-dependent single bump solutions numerically. In Section 4 we construct y-independent single bump solutions of (3) by means of a direct iteration scheme. Section 5 contains the conclusions and outlook.