Applied Mathematics and Computation 259 (2015) 379–389

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Sturmian theory for second order differential equations with mixed nonlinearities

A. Özbekler

Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey

ARTICLE INFO

Keywords:
Comparison
Leighton
Mixed nonlinear
Nonselfadjoint
Sturm–Picone
Wirtinger

ABSTRACT

In the paper, Sturmian comparison theory is developed for the pair of second order differential equations; first of which is the nonlinear differential equations

$$(m(t)y')' + s(t)y' + \sum_{i=1}^{n} q_i(t)|y|^{\alpha_i - 1}y = 0, \tag{1}$$

with mixed nonlinearities $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n$, and the second is the nonselfadjoint differential equations

$$(k(t)x')' + r(t)x' + p(t)x = 0. \tag{2}$$

Under the assumption that the solution of Eq. (2) has two consecutive zeros, we obtain Sturm–Picone type and Leighton type comparison theorems for Eq. (1) by employing the new nonlinear version of Picone's formula that we derive. Wirtinger type inequalities and several oscillation criteria are also attained for Eq. (1). Examples are given to illustrate the relevance of the results.

1. Introduction

In this paper we are concerned with Sturmian type comparison of solutions of nonselfadjoint Eq. (2) and nonlinear equations of the form (1) where k, r, p, m, s and q_i's are continuous functions on $[0, \infty)$. We assume without further mention that the functions $k(t), m(t)$ and $q_i(t), i = 1, \ldots n$, are positive and nonlinearities in Eq. (1) satisfy

$$\alpha_1 > \ldots > \alpha_m > 1 > \alpha_{m+1} > \ldots > \alpha_n. \tag{3}$$

By a solution $x(t)$ of Eq. (2) on an interval $J \subset [t_0, \infty)$ we mean a nontrivial continuous function defined on J with $k(t)x' \in C^1(J)$ such that $x(t)$ satisfies Eq. (2). A solution $y(t)$ of Eq. (1) is defined in a similar manner.

It is well-known that the Sturmian theory plays an important role in the study of qualitative behavior of solutions of linear, half-linear and nonlinear equations. Sturmian type comparison theorems for linear equations are very classical and well-known (see [1–10] and the references therein). In recent years, although the oscillation theory of nonlinear differential equations has been developed very rapidly, there are only a few papers with regard to the oscillation of their solutions as far as the Sturmian theory is concerned. Some pioneering works showed that there is a striking similarity between linear and half-linear [11,12], forced super-linear [13], forced quasilinear [14], nonlinear equations [15,16], linear and half-linear impulsive differential equations [17,18], showing that many results in the Sturmian comparison and oscillation theory for linear

E-mail addresses: aozbekler@gmail.com, aozbekler@atilim.edu.tr

http://dx.doi.org/10.1016/j.amc.2015.03.001
0096-3003/

equations can be carried to equations of these types. Motivated by this, we attempt to obtain analogous comparison results for the pair of second order differential equations of the form (1) and (2).

The proof of the well-known Sturm–Picone comparison theorem [1] (see also [7–9]) for the equations

$$(k(t)x')' + p(t)x = 0 \quad \text{and} \quad (m(t)y')' + q(t)y = 0,$$

is based on employing the Picone's identity

$$\frac{x}{y}(ykx' - xmy'|_a^b = \int_a^b \left[(k-m)(x'^2 + (q-p)x^2 + m\left(x' - \frac{x}{y}y'\right)2\right]dt, \tag{4}$$

which holds for all real valued functions x and y defined on an interval $[a,b]$ such that x, y, kx' and my' are differentiable on $[a,b]$ and $y \neq 0$ for $t \in [a,b]$. The formula (4) has also been used for establishing Wirtinger type inequalities for solutions of ordinary differential equations [5,7], and generalized to second order linear equations with damping terms [7, p.11] i.e. non-selfadjoint equations (2) and

$$(m(t)y')' + s(t)y + q(t)y = 0, \tag{5}$$

by employing the Picone type identity

$$\frac{x}{y}(ykx' - xmy')|_a^b = \int_a^b \left[(k-m)\left\{x' + \frac{(s-r)}{2(k-m)}x\right\}^2 + \frac{m}{y^2}\left(x'y - xy' - \frac{s}{2m}xy\right)^2 + \left\{q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right\}x^2\right]dt, \tag{6}$$

under the condition

$$k(t) \neq m(t) \quad \text{whenever} \quad r(t) \neq s(t) \quad \text{for all} \quad t \in I_0, \tag{C}$$

for some nondegenerate interval I_0. It is well known that condition (**C**) is crucial in obtaining a Picone's formula. If (**C**) fails to hold then Sturm–Picone, Leighton and Wirtinger type results require employing a so called "device of Picard"[2] (see also [7, p. 14]). We will show how this is possible for differential Eqs. (2) and (1) as well.

In 2005, Zhuang and Wu [16] establish some differential inequalities and then derived some Sturm type comparison theorems for Eq. (2) and nonlinear differential equations of the form

$$(m(t)y')' + s(t)y + q(t)f(y) = 0, \tag{7}$$

under the assumption that the functions r and s are differentiable and that $uf(u) > 0$ and $f'(u) \geqslant m > 0$ for $u \neq 0$. The comparison results given in [16] obtained by a kind of differential inequalities depending on the parameter "m"the lower bound for $f(.)$. In the first part of this paper, we do not impose any restriction on the differentiability of the functions r and s.

It is clear that a special case of Eq. (1) is the Emden–Fowler equation

$$(a(t)y')' + b(t)\mid y\mid^{\gamma-1}y = 0, \quad \gamma > 0. \tag{8}$$

When $\gamma \in (0,1)$, Eq. (8) is the sub-linear and when $\gamma \in (1,\infty)$, it is known as the super-linear equation. Last 50 years, the oscillation of Eq. (8) has been an increasing interest (see the book by Agarwal, et al. [19]). As far as the oscillation of more general equation

$$(a(t)y')' + b(t)G(y) = 0, \tag{9}$$

is considered, most of the results on oscillation of Eq. (8) are viable under the condition that $uf(u) > 0, u \neq 0$, and f satisfies some certain conditions of superlinearity and sublinearity, see [19–24] and references therein.

In 2007, Sun and Wong obtained an oscillation result [25, Theorem 2] for Eq. (1) by using the interval technique; They use a handy Lemma [25, Lemma 1] (see also Lemma 2.1) and Riccati technique to prove their results. In this paper, we make a new approach to the result of Sun and Wong [25] by using Lemma 1 in [25] and the Picone type identity that we derive.

The purpose of this paper is to show how Picone's formula can be used to extend the classical Sturmian theory to nonlinear equations of the form (1). Moreover, we also show that Picone's formula has also been used for proving Leighton type comparison results and setting Wirtinger type inequalities. By applying the comparison results, several oscillation criteria are established and examples are given to illustrate the importance of the results. Our results improve those in [7,16, pp. 11], bring in two alternative oscillation criteria to the result given in [25, Theorem 2].

2. Main results

Let (**C**) be satisfied. Suppose that x and y are continuous functions defined on I_0 such that $x'y' \in C(I_0)$ and $kx', my' \in C^1(I_0)$. If $y(t) \neq 0$ for any $t \in I_0$, then we may define

$$w(t) = \frac{x(t)}{y(t)}[y(t)k(t)x'(t) - x(t)m(t)y'(t)] \quad \text{for} \quad t \in I_0. \tag{10}$$

For clarity we suppress the variable t. In view of (2) and (1) it is not difficult to see, cf. [7] that