Applied Mathematics and Computation 259 (2015) 124–131

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Explicit form of parametric polynomial minimal surfaces with arbitrary degree

Gang Xu [a,*], Yaguang Zhu [a], Guozhao Wang [b], André Galligo [c], Li Zhang [a], Kin-chuen Hui [d]

[a] Department of Computer Science, Hangzhou Dianzi University, Hangzhou 310018, PR China
[b] Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China
[c] University of Nice Sophia-Antipolis, 06108 Nice Cedex 02, France
[d] Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Hong Kong

ARTICLE INFO

Keywords:
Minimal surface
Parametric polynomial minimal surface
Enneper surface
Conjugate minimal surface

ABSTRACT

In this paper, from the viewpoint of geometric modeling in CAD, we propose an explicit parametric form of a class of polynomial minimal surfaces with arbitrary degree, which includes the classical Enneper surface for the cubic case. The proposed new minimal surface possesses some interesting properties such as symmetry, containing straight lines and self-intersections. According to the shape properties, the proposed minimal surface can be classified into four categories with respect to $n = 4k - 1$, $n = 4k$, $n = 4k + 1$ and $n = 4k + 2$, where n is the degree of the coordinate functions in the parametric form of the minimal surface and k is a positive integer. The explicit parametric form of the corresponding conjugate minimal surface is given and the isometric deformation is also implemented.

1. Introduction

Minimal surface is a surface with vanishing mean curvature [5]. As mean curvature is the variation of area functional, minimal surfaces include those surfaces with minimum area and fixed boundaries [2]. Because of their attractive properties, minimal surfaces have been extensively employed in areas such as architecture, material science and ship manufacturing.

Parametric polynomial representation is a standard form widely used in Computer-aided Design. For parametric polynomial minimal surface, Enneper surface is the unique cubic parametric polynomial minimal surface. There are few research work on the parametric form of polynomial minimal surface with higher degree. Weierstrass representation is a classical parameterization of minimal surfaces. However, two functions have to be specified to construct the parametric form in Weierstrass representation. The Plateau-Bézier problems are investigated in [4], in which the area function is approximated by Dirichlet energy. By using geometric PDE method, the Plateau-Bézier or Plateau-B-spline problems are studied in [9]. The modeling methods of minimal subdivision surfaces are proposed in [6]. Some examples of parametric polynomial minimal surface of lower degree is proposed in [7,8]. Hao et.al studied a method to determine the quasi-Bézier surface of minimal area among all the quasi-Bézier surfaces with prescribed borders [1]. Li et al. studied the construction of approximate minimal surface with geodesic constraints [3].

* Corresponding author.
E-mail addresses: xugangzju@gmail.com (G. Xu), wanggz@zju.edu.cn (G. Wang), galligo@unice.fr (A. Galligo), kchui@mae.cuhk.edu.hk (K.-c. Hui).

http://dx.doi.org/10.1016/j.amc.2015.02.065
0096-3003/

In this paper, from the viewpoint of geometric modeling, we discuss the answer to the following questions: what are the possible explicit parametric form of polynomial minimal surface of arbitrary degree and how about their properties? The proposed minimal surfaces include the classical Enneper surface for cubic case, and have some interesting properties such as symmetry, containing straight lines and self-intersections. According to the shape properties, the proposed minimal surface can be classified into four categories with respect to $n = 4k - 1,\ n = 4k, n = 4k + 1$ and $n = 4k + 2$, where n is the degree of the coordinate functions in the parametric form of the minimal surface and k is a positive integer. The explicit parametric form of the corresponding conjugate minimal surfaces is given and the isometric deformation is also implemented.

2. Preliminary

In this section, we review some concepts and results related to minimal surfaces [5].

If the parametric form of a regular patch in $\mathbb{R}^3$ is given by

$$\boldsymbol{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad u, \quad v \in \Omega,$$

in which Ω is the parametric domain. Then the coefficients of the first fundamental form of $\boldsymbol{r}(u, v)$ are

$$E = \langle \boldsymbol{r}_u, \boldsymbol{r}_u \rangle, \quad F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle, \quad G = \langle \boldsymbol{r}_v, \boldsymbol{r}_v \rangle,$$

where $\boldsymbol{r}_u,\ \boldsymbol{r}_v$ are the first-order partial derivatives of $\boldsymbol{r}(u, v)$ with respect to u and v respectively and $\langle , \rangle$ defines the dot product of the vectors. The coefficients of the second fundamental form of $\boldsymbol{r}(u, v)$ are

$$L = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uu}), \quad M = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uv}), \quad N = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{vv}),$$

where $\boldsymbol{r}_{uu},\ \boldsymbol{r}_{vv}$ and $\boldsymbol{r}_{uv}$ are the second-order partial derivatives of $\boldsymbol{r}(u, v)$ and $(, ,)$ denotes the mixed product of the vectors. Then the mean curvature H and the Gaussian curvature K of $\boldsymbol{r}(u, v)$ are

$$H = \frac{EN - 2FM + LG}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Definition 2.1. If $\boldsymbol{r}(u, v)$ satisfies $E = G,\ F = 0$ for all $u,\ v \in \Omega$, then $\boldsymbol{r}(u, v)$ is called surface with isothermal parameterizations.

Definition 2.2. If $\boldsymbol{r}(u, v)$ satisfies $\boldsymbol{r}_{uu} + \boldsymbol{r}_{vv} = 0$ for all $u,\ v \in \Omega$, then $\boldsymbol{r}(u, v)$ is called harmonic surface.

Definition 2.3. If $\boldsymbol{r}(u, v)$ satisfies $H = 0$ for all $u,\ v \in \Omega$, then $\boldsymbol{r}(u, v)$ is called minimal surface.

Lemma 2.4. *A surface with isothermal parameterization is a minimal surface if and only if it is a harmonic surface.*

Proof. Suppose that $\boldsymbol{r}(u, v)$ is a surface with isothermal parameterization, that is, $\boldsymbol{r}(u, v)$ satisfies $E = G$ and $F = 0$. Then we have,

$$H = \frac{EN - 2FM + LG}{2(EG - F^2)} = \frac{N + L}{2E} = \frac{(\boldsymbol{r}_u \times \boldsymbol{r}_v)(\boldsymbol{r}_{uu} + \boldsymbol{r}_{vv})}{2E}.$$

As $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$ and $E = (\boldsymbol{r}_v, \boldsymbol{r}_u) \neq 0$, hence $H = 0$ if and only if $\boldsymbol{r}_{uu} + \boldsymbol{r}_{vv} = 0$. Thus, the proof is completed. □

Definition 2.5. If two differentiable functions $p(u, v),\ q(u, v) : U \mapsto \mathbb{R}$ satisfy the Cauchy–Riemann equations

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial v}, \quad \frac{\partial p}{\partial v} = -\frac{\partial q}{\partial u}$$

and both are harmonic, then the functions are said to be harmonic conjugate.

Definition 2.6. If $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ are with isothermal parameterizations such that p_k and q_k are harmonic conjugate for $k = 1,\ 2,\ 3$, then P and Q are said to be parametric conjugate minimal surfaces.

For example, helicoid and catenoid are a pair of conjugate minimal surface. A pair of conjugate minimal surfaces satisfy the following lemma.

Lemma 2.7. *Given two conjugate minimal surfaces P and Q and a real number t, all surfaces of the one-parameter family*