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The least squares anti-bisymmetric solution and the optimal approximation solution for Sylvester equation

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ABSTRACT

In this paper, a modified conjugate gradient iterative method for solving Sylvester equation is presented. By using this iterative method, the least squares anti-bisymmetric solution and the optimal approximation solution can be obtained. Here we present the derivation and theoretical analysis of our iterative method. Numerical results illustrate the feasibility and effectiveness of the proposed iterative method.

1. Introduction

As we know, in the field of matrix algebra, the Sylvester matrix equation is one kind of very important matrix equation. The study of Sylvester matrix equation is of great importance in the control theory and application, such as pole assignment, the observer design and construct Lyapunov function and so on. A large number of papers have been written for solving the various types of Sylvester matrix equations [1–20].

In [21], Don presented the general symmetric solution to matrix equation $AX = B$ by applying a formula for the partitioned minimum-norm reflexive generalized. In [22], the symmetric solution of linear matrix equation like $AX = B$ had been considered using the singular-value, generalized singular-value, real Schur, and real generalized Schur decompositions. After these works, there were some results about solvability conditions, solution formulas and least-squares approximate solutions for more general forms of the linear matrix equations. For example, Liao and Bai in [23] studied the symmetric positive semidefinite solution of the matrix equation $AX_1A^T + BX_2B^T = C$ and the bisymmetric positive semidefinite solution of the matrix equation $D^TXD = C$. They in [24] considered the least-squares solution of $AXB = D$ with respect to symmetric positive semidefinite matrix X. In [25] they and Lei presented an algorithm for finding the best approximate solution of matrix equation $AXB + CYD = E$. Wang in [26] considered the bisymmetric solution of the real quaternion matrix equation $AX = B$. In [27], Zhao et al. got the bisymmetric least squares solution under a central principal submatrix constraints of matrix equation $AX = B$. In [28], Sheng et al. gave the conditions for the existence of the anti-question of the anti-bisymmetric matrix. The anti-bisymmetric solution of the matrix equation $AX = B$ had been derived by the generalized singular value decomposition in [29]. In [30], Li et al. presented the least squares anti-bisymmetric solution of the matrix equation $AX = B$ by special transformation and approximating disposal. Its iterative method was on the base of conjugate gradient method. Using the thinking methods of the literature [30] for reference, we present an iterative method to solve the least squares anti-bisymmetric solution of the Sylvester matrix equation $AX + XB = C$.

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In this paper, we mainly consider the following problems:

Problem I Given $n \times n$ real matrices A, B and C, find an $n \times n$ anti-bisymmetric matrix X such that

$$\|AX + XB - C\| = \min.$$

Problem II Given an $n \times n$ real matrix X^*, find $\hat{X} \in S_E$ such that

$$\|\hat{X} - X^*\| = \min_{X \in S_E}(\|X - X^*\|),$$

where $\|\cdot\|$ is Frobenius norm and S_E is the solution set of Problem I.

The organization of this paper is as follows. In Section 2, we get the equivalent transformation of Problem I. In Section 3, we present an iterative method for solving Problem I. In Section 4, we obtain the solution of Problem II using the iterative method proposed in Section 3. Numerical results are given in Section 5 to show feasibility and effectiveness of our iterative method. Finally, in Section 6, we draw a brief conclusion and make some remarks.

The notations and symbols used in this paper are summarized as follows. Let $R^{n\times n}$ and $BASR^{n\times n}$ be the set of $n \times n$ real matrices and the set of $n \times n$ real anti-bisymmetric matrices. The symbols $A^T, r(A), R(A)$ and $\mathrm{tr}(A)$ respectively stand for the transpose, rank, column space and trace of matrix A. $\langle A, B\rangle = \mathrm{tr}(B^TA)$ is defined as the inner product of the two matrices, which generates the Frobenius norm, i.e. $\|A\|_F^2 = \langle A, A\rangle = \mathrm{tr}(A^TA)$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. The vec operator is represented as $\mathrm{vec}(\cdot)$ and $\mathrm{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$, where a_k is the kth column of A. I_n and e_j represent the identity matrix of order n and the jth column of I_n, respectively. The $n \times n$ reverse unit matrix $S = (e_n, e_{n-1}, \ldots, e_1)$. It's simple to verify that $S^2 = I_n, S = S^T = S^{-1}$. An $n \times n$ matrix X is called an anti-bisymmetric matrix if $X = -X^T$ and $X = SXS$.

2. The equivalent transformation of Problem I

In this section, we firstly review two lemmas which will be used in what follows.

Lemma 2.1. *If matrix $X \in BASR^{n\times n}$, then $X + SXS \in BASR^{n\times n}$.*

Lemma 2.2 [31]. *Suppose that $A \in R^{m\times n}, x \in R^{n\times 1}, b \in R^{m\times 1}$, then finding the least norm solution of the linear equation $Ax = b$ is equivalent to finding the solution of the linear equation $A^TAx = A^Tb$.*

For the convenience of discussion, we introduce marks as follows:

$$F(X) = (A^TA + BB^T)X + X(A^TA + BB^T) + (AXB^T + A^TXB) - (AXB^T + A^TXB)^T + S\left[(A^TA + BB^T)X + X(A^TA + BB^T)\right]S$$
$$+ S\left[(AXB^T + A^TXB) - (AXB^T + A^TXB)^T\right]S,$$

$$H = A^TC + CB^T - (A^TC + CB^T)^T + S\left[A^TC + CB^T - (A^TC + CB^T)^T\right]S.$$

Obviously, $H \in BASR^{n\times n}$.

Theorem 2.1. *Problem I is equivalent to finding the anti-bisymmetric solution of the matrix equation*

$$F(X) = H, \tag{2.1}$$

and it is always consistent.

Proof. According to the definition of anti-bisymmetric matrix and the properties of Frobenius norm, it follows that

$$\begin{aligned}\min_{X\in BASR^{n\times n}}(\|AX+XB-C\|^2) &= \frac{1}{4}\min_{X\in BASR^{n\times n}}\left(\|AX+XB-C\|^2 + \|XA^T+B^TX+C^T\|^2 + \|ASXS+SXSB-C\|^2 + \|SXSA^T+B^TSXS+C^T\|^2\right)\\ &= \frac{1}{4}\min_{X\in BASR^{n\times n}}\left\|\begin{pmatrix}AX+XB-C\\ XA^T+B^TX+C^T\\ ASXS+SXSB-C\\ SXSA^T+B^TSXS+C^T\end{pmatrix}\right\|^2 = \frac{1}{4}\min_{X\in BASR^{n\times n}}\left\|\begin{pmatrix}AX+XB\\ XA^T+B^TX\\ ASXS+SXSB\\ SXSA^T+B^TSXS\end{pmatrix} - \begin{pmatrix}C\\ -C^T\\ C\\ -C^T\end{pmatrix}\right\|^2.\end{aligned} \tag{2.2}$$

Using vec operator, we have that Problem I is equivalent to the following minimum residual problem