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Solving a class of nonlinear matrix equations via the coupled fixed point theorem

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ABSTRACT

We consider a class of nonlinear matrix equations of the type

$$X = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i - \sum_{j=1}^{k} B_j^* \mathcal{K}(X) B_j, \tag{1}$$

where Q is a positive definite matrix, A_i, B_j are arbitrary $n \times n$ matrices and $\mathcal{G}, \mathcal{K}$ are two order-preserving or order-reversing continuous maps from $\mathcal{H}(n)$ into $\mathcal{P}(n)$. In this paper we first discuss existence and uniqueness of coupled fixed points in a L-space endowed with reflexive relation. Next on the basis of the coupled fixed point theorems, we prove the existence and uniqueness of positive definite solutions to such equations.

1. Introduction

The existence of some new fixed point theorems for contractive maps in partially ordered metric spaces has been consider recently. The first result in this direction was given by Ran and Reurings [1], where they extended the Banach contraction principle in partially ordered sets and presented some applications to matrix equations. For more details about general nonlinear matrix equations we also refer to the articles [2–6]. Also, existence and uniqueness of fixed point in ordered L-spaces has been considered in [7–9], where some applications to matrix equations are presented. During the last few decades, the discussion on coupled fixed point theorems has been growing rapidly because of their important role in the study of nonlinear differential equations, nonlinear integral equations and nonlinear matrix equations. In [10], Bhaskar and Lakshmikantham established some coupled fixed theorems in ordered metric spaces for mappings having the mixed monotone property and satisfying a given contractive condition. In this paper, we discuss some results on the existence and uniqueness of coupled fixed points in L-spaces endowed with a reflexive relation and some applications to nonlinear matrix equations. In order to apply coupled fixed point results, we use from coupled fixed points of the nonlinear map:

$$\mathcal{F} : \mathcal{H}(n) \times \mathcal{H}(n) \to \mathcal{H}(n), \qquad \mathcal{F}(X, Y) = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i - \sum_{j=1}^{k} B_j^* \mathcal{K}(Y) B_j, \tag{2}$$

with respect to partial order $\leqslant$ on $\mathcal{H}(n)$. where $\mathcal{H}(n)$ is the set of all $n \times n$ Hermitian matrices and Q is a positive definite matrix, A_i, B_j are arbitrary $n \times n$ matrices and $\mathcal{G}, \mathcal{K}$ are two order-preserving or order-reversing continuous maps from

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$\mathcal{H}(n)$ into $\mathcal{P}(n)$ such that $\mathcal{G}(0) = \mathcal{K}(0) = 0$. In the rest of this introduction we will briey recall the definitions and basic properties of L-spaces. For more informations we refer to [11,7,12].

Definition 1.1. Let X be a nonempty set. Denote $s(X) = \{\{x_n\}_{n\in\mathbb{N}} : x_n \in X,\ n \in \mathbb{N}\}$. Suppose that $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ a mapping. Then the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [13]) if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}} \in c(x)$ and $Lim(\{x_n\}_{n\in\mathbb{N}}) = x$.
(ii) If $\{x_n\}_{n\in\mathbb{N}} \in c(x)$ and $Lim(\{x_n\}_{n\in\mathbb{N}}) = x$, then for all subsequences, $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ we have that $\{x_{n_k}\}_{k\in\mathbb{N}} \in c(X)$ and $Lim(\{x_{n_k}\}_{k\in\mathbb{N}}) = x$.

The elements of $c(X)$ are called convergent sequences and $Lim(\{x_n\}_{n\in\mathbb{N}})$ is the limit of the sequence, also written $x_n \to x$ as $n \to \infty$. An L-space is denoted by $(X, \to)$.

Notation 1.2. Let X be a nonempty set and let $f : X \times X \to X$ be a mapping. We define $f^0(x,y) = x$ and $f^n(x,y) = f\left(f^{n-1}(x,y), f^{n-1}(y,x)\right)$ for all $x, y \in X, n \in \mathbb{N}$.

Definition 1.3. Let X be a nonempty set and let $f, g : X \times X \to X$ be two map. Then

(i) The Cartezian product of f and g is denoted by $f \times g$, and it is defined in the following way:

$$f \times g(x,y) = (f(x,y), g(y,x)).$$

(ii) An element $(x,y) \in X \times X$ is called a coupled fixed point of f, if $f(x,y) = x$ and $f(y,x) = y$ and an element $x \in X$ is called a fixed point of f, if $f(x,x) = x$. We will denote the set of all the coupled fixed points of f by F_f^c and the set of all the fixed points of f by F_f.
(iii) An element $(x,y) \in X \times X$ is called a coupled attractor basin element of f with respect to $(x^*, y^*) \in X \times X$, if $f^n(x,y) \to x^*$ and $f^n(y,x) \to y^*$, as $n \to \infty$ and an element $x \in X$ is called an attractor basin element of f with respect to $x^* \in X$, if $f^n(x,x) \to x^*$, as $n \to \infty$. We will denote the set of all the coupled attractor basin elements of f with respect to (x^*, y^*) by $A_f^c(x^*, y^*)$ and the set of all the attractor basin elements of f with respect to $x^* \in X$ by $A_f(x^*)$.
(iv) The mapping f is called orbitally continuous if $(x,y), (a,b) \in X \times X$ and $f^{n_k}(x,y) \to a, f^{n_k}(y,x) \to b$, as $k \to \infty$ imply $f^{n_k+1}(x,y) \to f(a,b)$ and $f^{n_k+1}(y,x) \to f(b,a)$ as $k \to \infty$.
(v) The mapping f is called a Picard operator, if there exists $x^* \in X$ such that:
(1) $F_f = \{x^*\}$.
(2) $A_f(x^*) = X$.

Also f is called a weakly Picard operator, if the sequences $\{f^n(x,x)\}_{n\in\mathbb{N}}$ converge for all $x \in X$ and the limits (which may depend on x) are a fixed points of f.

Definition 1.4. An ordered L-space is a triple $(X, \to, \leqslant)$ such that

(i) $(X, \to)$ is an L-space.
(ii) $(X, \leqslant)$ is a partially ordered set.
(iii) If $x_n \to x, y_n \to y$ as $n \to \infty$ and $x_n \leqslant y_n$ for all $n \in \mathbb{N}$, then $x \leqslant y$.

Let (X, d) be a metric space, and consider $c(X)$ the family of all d-convergent sequences and Lim is the mapping that sends every d-convergent sequence into its limit. Thus every metric space is a L-space. Therefore, a mapping $f : X \times X \to X$ is Picard mapping if there exists $x^* \in X$ such that $F_f = \{x^*\}$ and $f^n(x,x) \to x^*$, as $n \to \infty$ for all $x \in X$.

Example 1.5. Let (X, d) be a complete metric space, and $f : X \times X \to X$ is a k-contraction, i.e., there exists a $k \in]0, 1[$ with:

$$d(f(x,y), f(u,v)) \leqslant \frac{k}{2}[d(x,u) + d(y,v)], \quad \forall x, y, u, v \in X.$$

Then by induction we have

$$d(f^{n+1}(x,x), f^n(x,x)) \leqslant k^n d(f(x,x), x).$$