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Iterative approximation to a coincidence point of two mappings

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ABSTRACT

In this article two methods for approximating the coincidence point of two mappings are studied and moreover, rates of convergence for both methods are given. These results are illustrated by several examples, in particular we apply such results to study the convergence and their rate of convergence of these methods to the solution of a nonlinear integral equation and of a nonlinear differential equation.

1. Introduction

Some nonlinear problems arising from many areas of the applied sciences reduce, under the mathematical point of view, to solving a given equation of the form

$$\text{Find } x \in X \quad \text{such that} \quad Tx = Sx, \tag{1}$$

where X is a nonempty set, Y is a Banach space, and $T, S : X \to Y$ are two mappings.

It is well known that the existence of a solution to problem (1) is, under appropriate conditions, equivalent to the existence of a fixed point for a certain mapping. In this sense, R. Machuca [18] proved a coincidence theorem by using Banach's contraction principle. This same principle was used by K. Goebel [12] to obtain a similar result under much weaker assumptions. Recently, several extensions of the above results due to Machuca and Goebel, as well as some application to the existence of solutions for various types of functional equations, have been obtained, for instance see [2,3,5,6,9,10].

Several physical problems, expressed as a coincidence point equation $Tx = Sx$, are solved by an approximating sequence $(x_n) \subseteq X$ generated by an iterative procedure $f(T, S, x_n)$. Let the sequence (x_n) converge to a coincidence point of T and S. Jungck [15] introduced the following iterative scheme: given $x_1 \in X$, there exists a sequence (x_n) in X such that $Tx_{n+1} = Sx_n$. This procedure becomes the Picard iteration when $X = Y$ and $T = I_d$, where I_d is the identity map on X, in this article the author proved that if (X, d) and (Y, ρ) are two complete metric spaces and T and S satisfy both that $S(X) \subseteq T(X)$ and that for every $x, y \in X$ the inequality $d(Sx, Sy) \leqslant \kappa\, d(Tx, Ty)$ with $0 \leqslant \kappa < 1$ holds, then (x_n) converges to the unique coincidence point of T and S. Later, this type of convergence results were generalized for more general classes of contractive type mappings, see [1,4,6] (to see another type of iterative schemes we can quote [19]).

Let $(X, \|\cdot\|)$ be a real Banach space and C be a nonempty closed convex subset of X. Recall that a mapping $T : C \to C$ is said to be nonexpansive on C if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. In 1967, Halpern [13] firstly introduced the following explicit iterative scheme in Hilbert spaces:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Tx_n, \quad \text{for each} \quad n \in \mathbb{N}, \tag{2}$$

where (α_n) is a sequence in $(0, 1)$ and $u \in C$ is fixed.

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He pointed out that the control conditions $(A_1)\ \sum_{n\in\mathbb{N}}\alpha_n = \infty$ and $(A_2)\ \lim_{n\to\infty}\alpha_n = 0$ are necessary for the convergence of the iterative scheme (2) to a fixed point of T. In 1992, Wittmann [23] proved a strong convergence result in Hilbert spaces for the iterative method (2) under conditions $(A_1), (A_2)$ and $(A_3)\ \sum_{n\in\mathbb{N}}|\alpha_n - \alpha_{n+1}| < \infty$. Shioji and Takahashi [21], in 1997, extended Wittmann's result to a reflexive Banach space with a uniformly Gâteaux differentiable norm and in which each nonempty closed convex and bounded subset enjoys the fixed point property for nonexpansive mappings.

The purpose of this paper is to establish sufficient conditions for the convergence of Picard–Jungck's scheme as well as the following algorithm: Let $(\alpha_n)_{n\in\mathbb{N}}$ be a real sequence in $(0,1)$. For an arbitrarily initial $x_1 \in X$, define a sequence $(x_n)_{n\in\mathbb{N}}$ recursively by the following implicit scheme:

$$Tx_{n+1} = \alpha_n Tx_1 + (1-\alpha_n)Sx_n, \quad \text{for each} \quad n \in \mathbb{N}, \tag{3}$$

where $T(X)$ is a convex closed subset of Y and $S(X) \subseteq T(X)$. Notice that, this procedure becomes the Halpern iteration when $X = Y$ and $T = I_d$.

2. Notations and preliminaries

In this section we shall recall some definitions and results that are needed later on.

Throughout this article $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all non-negative real numbers and the set of all positive integer numbers respectively.

Definition 2.1. Let X and Y be two nonempty sets and $T, S : X \to Y$ two mappings. If there exists $x \in X$ such that $Sx = Tx$ then x is called a *coincidence point* of S and T. In the case that $X = Y$, if $Sx = Tx = x$, then x is called a *common fixed point* of S and T.

Definition 2.2. Let S and T be two self-mappings of a nonempty set X. The pair of mappings S and T is said to be *weakly compatible* if they commute at their coincidence points, that is, $TSx = STx$ whenever $Tx = Sx$.

The following straightforward result states a relationship between coincidence points and common fixed points of two weakly compatible mappings, see Proposition 1.4 in [1].

Lemma 2.1. *Let S and T be weakly compatible self-mappings of a nonempty set X. If S and T have a unique coincidence point x, then x is the unique common fixed point of S and T.*

There are a number of generalizations of metric spaces. One such generalization is the one given by semi-metric spaces initiated by Fréchet [7].

Definition 2.3. A semi-metric space (X, d) is a nonempty set X endowed with a function, called semi-metric, $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(a) $d(x,y) = 0$ if, and only if, $x = y$
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$

Notice that every metric space (or, more general, every quasi-metric space [14]) is semi-metric but not conversely.

By $\mathcal{F}$ denote the set of all monotone nondecreasing real functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(t) = 0$ if and only if $t = 0$.

Let (X, d) and (Y, ρ) be two semi-metric spaces and let $T : X \to Y$ be a mapping.

(a) T is said to be φ-expansive if there exists a function $\varphi \in \mathcal{F}$ such that

$$\rho(Tx, Ty) \geqslant \varphi(d(x,y)), \quad \text{for all} \quad x, y \in X.$$

(b) If φ is the identity mapping, then T is called expansive.
(c) T is a ϕ-contraction if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (i.e.; if ϕ is nondecreasing and the iterate $\phi^k(t) \to 0$, as $k \to +\infty$) and $\rho(Tx, Ty) \leqslant \phi(d(x,y))$, for all $x, y \in X$.

The following results will be used in order to study the convergence of the Picard–Jungck iterative method. Both results can be found in [10].

Theorem 2.1 [10, Theorem 3.12]. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that:*

(i) *$T : X \to Y$ is an expansive mapping,*
(ii) *the mapping $S : X \to Y$ is a ϕ-contraction,*
(iii) $S(X) \subseteq T(X)$.