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The distribution of zeros of all solutions of first order neutral differential equations

Faroq A. Baker, Hassan A. El-Morshedy *

Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt

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ABSTRACT

This paper is concerned with the distribution of zeros of all solutions of the first-order neutral differential equation

$$[x(t) + p(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0, \quad t \geqslant t_0,$$

where

$$p \in C([t_0,\infty),[0,\infty)), \quad Q \in C([t_0,\infty),(0,\infty)) \text{ and } \tau,\sigma \in R^+.$$

New estimations for the distance between adjacent zeros of this neutral equation are obtained via comparison with a corresponding differential inequality. These results extend some known results from the non-neutral to the neutral case and improve other published results as well.

1. Introduction

The oscillation theory of neutral differential equations has been investigated extensively in the last decades and increasingly attract much interest due to the valuable applications of these type of equations in many fields; see [5–8]. This theory, generally, investigates the problem of existence of an infinite number of zeros of all solutions; see [1–3,6] for the recent advances in oscillation theory. A crucial question in the theory is to determine the zeros locations for each solution of a given equation. This problem, for functional differential equations, did not receive the deserved interest, although it gives more insight into the properties of the solutions which means better understanding for some phenomenon that can be modeled by these equations.

This work is devoted to study the distribution of zeros of all solutions of the neutral equation

$$[x(t) + p(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0, \quad t \geqslant t_0, \tag{1.1}$$

where $p \in C([t_0,\infty),[0,\infty)), Q \in C([t_0,\infty),(0,\infty))$ and $\tau,\sigma \in R^+$ such that $\sigma > \tau$. As far as these authors know; it seems that [9,10,12,13] are the known published papers that deal with the distribution of zeros of neutral equations of the form (1.1).

The usual utilized technique is to relate the distance between adjacent zeros of any solution $x(t)$ of (1.1) to a positivity problem of certain solution of a first order delay differential inequality

$$x'(t) + P(t)x(t-r) \leqslant 0, \tag{1.2}$$

* Corresponding author.
E-mail address: elmorshedy@yahoo.com (H.A. El-Morshedy).

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on some bounded real intervals, where $P \in C([t_0, \infty), [0, \infty))$ depends on the coefficients in (1.1) while the delay $r > 0$ depends on σ and τ. We will be able to obtain new estimates for the distance between zeros of all solutions of (1.1). Also some known results from [9,10,12,13] will be improved. Moreover, our results produce oscillation criteria for Eq. (1.1) where a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros. Eq. (1.1) is called oscillatory if all its solutions are oscillatory.

The organization of this paper is as follows; in Section 2, we gather improved versions of some known results from [10,11] as well as some new ones based on new ideas from El-Morshedy [4]. These results give necessary conditions for the positivity of certain solution of the inequality (1.2) and play an important role in our main results. The purpose of this collection is to ease our calculations and avoid repetitions as well. Section 3 contains our results on the neutral Eq. (1.1) and some illustrative examples.

Throughout this work; $d_s(x)$ denotes the least upper bound of the distances between adjacent zeros of any solution $x(t)$ of Eq. (1.1) on $[s, \infty)$.

2. First order differential inequalities

If $x(t)$ is a solution of (1.2) which is positive on an interval $I \subset [0, \infty)$, the bounds of $\frac{x(t-\tau)}{x(t)}$ for $t \in I$ are used to obtain necessary condition for the positivity of $x(t)$ on I and hence play a fundamental role in our proofs. In [9–11] some such bounds are obtained using sequences depending on a number $\rho \geqslant 0$ satisfies that

$$\int_{t-r}^{t} P(s)ds \geqslant \rho, \quad t \geqslant t_0 + r. \tag{2.1}$$

In fact, Xianhua and Jianshe [11] define a sequence $\{f_n(\rho)\}$ when $0 < \rho < 1$, by

$$f_0(\rho) = 1, \quad f_1(\rho) = \frac{1}{1-\rho}, \quad f_{n+2}(\rho) = \frac{f_n(\rho)}{f_n(\rho) + 1 - e^{\rho f_n(\rho)}}, \quad n = 0, 1, \ldots \tag{2.2}$$

They showed, for $0 \leqslant \rho \leqslant \frac{1}{e}$, that $1 \leqslant f_n(\rho) \leqslant f_{n+2}(\rho) \leqslant e$ for $n \geqslant 0$ and $\lim_{n\to\infty} f_n(\rho) = f(\rho) \in [1, e]$, where $f(\rho)$ is a real root of the equation:

$$f(\rho) = e^{\rho f(\rho)}, \tag{2.3}$$

while, for $\rho > \frac{1}{e}$, it is proved that either $f_n(\rho)$ is nondecreasing and $\lim_{t\to\infty} f_n(\rho) = \infty$ or $f_n(\rho)$ is negative or $f_n(\rho) = \infty$ after finite number of terms.

Wu and Xu [10] defined a sequence $\{g_m(\rho)\}$ for $0 < \rho < 1$, by

$$g_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad g_{m+1}(\rho) = \frac{2\left(1 - \rho - \frac{1}{g_m(\rho)}\right)}{\rho^2}, \quad m = 1, 2, \ldots \tag{2.4}$$

and proved that $\{g_m(\rho)\}$ is decreasing. They found also that $g_{m+1}(\rho) > \frac{1-\rho}{\rho^2}$ for $m = 1, 2, \ldots$ when $0 < \rho \leqslant \frac{1}{e}$ and hence; there exists a function $g(\rho) = \frac{2\left(1-\rho-\frac{1}{g(\rho)}\right)}{\rho^2}$ such that $lim_{m\to\infty} g_m(\rho) = g(\rho)$.

In the next two lemmas, we prove an updated version of [11, Lemma 3] and [10, Lemma 3] respectively.

Lemma 2.1. *Assume that (2.1) holds for $\rho > 0$ and there exist $T_1 \geqslant t_0 + r, |\delta| \leqslant r, T \geqslant T_1 + (1+n)r - \delta$ and a function $x(t)$ satisfying inequality (1.2) on $[T_1, T]$ with $x'(t) \leq 0$ for $t \in [T_1 - \delta, T]$. If $x(t)$ is positive on $[T_1, T]$, then*

$$\frac{x(t-r)}{x(t)} \geqslant f_n(\rho) > 0, \quad \text{for } t \in [T_1 + (1+n)r - \delta, T], \tag{2.5}$$

for some integer $n \geqslant 0$, where $f_n(\rho)$ is defined by (2.2).

Proof. Since $x(t)$ is nonincreasing on $[T_1 - \delta, T]$, we find

$$\frac{x(t-r)}{x(t)} \geqslant 1 = f_0(\rho) \text{ for } t \in [T_1 + r - \delta, T]. \tag{2.6}$$

Integrating inequality (1.2) from $t - r$ to t, where $T_1 + 2r - \delta \leqslant t \leqslant T$, we obtain

$$x(t-r) \geqslant x(t) + \int_{t-r}^{t} P(s)x(s-r)ds \geqslant x(t) + \rho x(t-r).$$