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Inequalities and completely monotonic functions associated with the ratios of functions resulting from the gamma function

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ABSTRACT

We prove that some functions associated with the products $\prod_{k=1}^{n}\frac{2k-1}{2k}$, $\prod_{k=1}^{n}\frac{3k-2}{3k-1}$, $\prod_{k=1}^{n}\frac{3k-2}{3k}$ and $\prod_{k=1}^{n}\frac{3k-1}{3k}$ are completely monotonic. By using the results obtained, we derive some inequalities for the previous products. Our results improve some known inequalities.

1. Introduction

A function *f* is said to be completely monotonic on an interval *I* if it has derivatives of all orders on *I* and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad (x \in I;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1,2,3,\ldots\}). \tag{1.1}$$

Dubourdieu [6, p. 98] pointed out that, if a non-constant function *f* is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (1.1). See also [7] for a simpler proof of this result. It is known (Bernstein's Theorem) that *f* is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [14, p. 161]). The main properties of completely monotonic functions are given in [[14], Chapter IV]. We also refer to [2], where an extensive list of references on completely monotonic functions can be found.

The problem of finding new and sharp inequalities for the gamma function Γ and, in particular, for the Wallis ratio

$$\frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \tag{1.2}$$

has attracted the attention of many researchers (see [3,4,8–10,12] and references therein). Here, we employ the special double factorial notation as follows:

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!,$$

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1) = \pi^{-1/2} 2^n \Gamma\left(n + \frac{1}{2}\right),$$

$$0!! = 1, \quad (-1)!! = 1$$

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(see [1, p. 258]). For example, Chen and Qi [4] proved that for $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{\pi(n+\frac{4}{\pi}-1)}} \leqslant \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+\frac{1}{4})}}, \tag{1.3}$$

where the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. This inequality is a consequence on $(0,\infty)$ of the complete monotonicity of the function

$$V(x) = \frac{\Gamma(x+1)}{\sqrt{x+\frac{1}{4}}\,\Gamma(x+\frac{1}{2})} \tag{1.4}$$

(see [5]).

It is well known (see [1, p. 255]) that

$$\Gamma\left(n+\frac{1}{3}\right) = \frac{1\cdot 4\cdot 7\cdot 10\ldots(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right)$$

and

$$\Gamma\left(n+\frac{2}{3}\right) = \frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{3^n}\Gamma\left(\frac{2}{3}\right).$$

Motivated by the inequality (1.3), Mortici et al. [13] considered the following products for every integer $n \in \mathbb{N}$:

$$\prod_{k=1}^{n}\frac{3k-2}{3k} = \frac{1\cdot 4\cdot 7\ldots(3n-2)}{3\cdot 6\cdot 9\ldots(3n)} = \frac{\Gamma(n+\frac{1}{3})}{\Gamma(n+1)\Gamma(\frac{1}{3})} \tag{1.5}$$

and

$$\prod_{k=1}^{n}\frac{3k-1}{3k} = \frac{2\cdot 5\cdot 8\ldots(3n-1)}{3\cdot 6\cdot 9\ldots(3n)} = \frac{\Gamma(n+\frac{2}{3})}{\Gamma(n+1)\Gamma(\frac{2}{3})}. \tag{1.6}$$

Mortici et al. [[13], Theorems 1 and 2] proved that the functions

$$f(x) = \ln\frac{\left(\frac{1}{2\pi}\sqrt{3}\Gamma(\frac{2}{3})\right)^3}{x^2\left(\frac{\Gamma(x+\frac{1}{3})}{\Gamma(x+1)\Gamma(\frac{1}{3})}\right)^3} \quad \text{and} \quad g(x) = \ln\frac{\frac{1}{\Gamma^3(\frac{2}{3})}}{x\left(\frac{\Gamma(x+\frac{2}{3})}{\Gamma(x+1)\Gamma(\frac{2}{3})}\right)^3}$$

are completely monotone on $(0,\infty)$. As a result, the following inequalities:

$$\frac{\alpha}{n^{2/3}} \leqslant \frac{1\cdot 4\cdot 7\ldots(3n-2)}{3\cdot 6\cdot 9\ldots(3n)} < \frac{\beta}{n^{2/3}} \tag{1.7}$$

and

$$\frac{\sigma}{n^{1/3}} \leqslant \frac{2\cdot 5\cdot 8\ldots(3n-2)}{3\cdot 6\cdot 9\ldots(3n)} < \frac{\tau}{n^{1/3}} \tag{1.8}$$

are established, where the constants

$$\alpha = \frac{1}{3}, \quad \beta = \frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right) = 0.37328\ldots$$

and

$$\sigma = \frac{2}{3}, \quad \tau = \frac{1}{\Gamma(\frac{2}{3})} = 0.73848\ldots$$

are the best possible.

Mortici et al. [13, Theorems 3 and 4] presented further improvements of the inequalities (1.7) and (1.8) as follows:

$$p_n < \frac{1\cdot 4\cdot 7\ldots(3n-2)}{3\cdot 6\cdot 9\ldots(3n)} < q_n, \tag{1.9}$$

where

$$p_n = \frac{\frac{\sqrt{3}}{2\pi}\Gamma(\frac{2}{3})}{(n^2+\frac{1}{3}n)^{1/3}}\exp\left\{-\frac{2}{81n^2}\right\} \quad \text{and} \quad q_n = \frac{\frac{\sqrt{3}}{2\pi}\Gamma(\frac{2}{3})}{(n^2+\frac{1}{3}n)^{1/3}}\exp\left\{-\frac{2}{81n^2}+\frac{2}{243n^3}\right\}$$