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A family of methods for solving nonlinear equations

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ABSTRACT

We present a family of methods for solving nonlinear equations. Some well-known classical methods and their modifications belong to our family, for example Newton, Potra-Pták, Chebyshev, Halley and Ostrowski's methods. Convergence analysis shows that our family contains methods of convergence order from 2 to 4. All our fourth order methods are optimal in terms of the Kung and Traub conjecture. Several examples are presented and compared.

1. Introduction

In this paper we consider a family of iterative methods for finding a simple root α of the nonlinear equation $f(x) = 0$. We assume that f has sufficient number of continuous derivatives in a neighborhood of α and that f' does not vanish in the interval of interest.

Let us observe that many methods have a similar form, namely,

$$x_{n+1} = F(x_n), \quad n = 0, 1, \ldots \tag{1}$$

where

$$F(x) = x - u(x)\phi(x), \tag{2}$$

$$u(x) = \frac{f(x)}{f'(x)} \tag{3}$$

but only differ in the function ϕ, see [5,7–10,25–27].

Newton's method is a well-known iterative method for computing approximations of α by using (1)–(3) with $\phi(x) = 1$. The Newton's method converges quadratically in some neighborhood of α for some appropriate start value x_0.

To get a method with a higher order of convergence, some new variants of Newton's method have been proposed. In recent years, many modified iterative methods for solving nonlinear equations have been developed to improve the local order of convergence of some classical methods such as Newton, Potra-Pták, Chebyshev, Cauchy, Halley and Ostrowski's methods.

As the order of an iterative method increases, so does the number of function evaluations per step. The efficiency index [3] gives a measure of balance between those quantities, according to the formula $p^{1/n}$, where p is the order of the method and n

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the number of function evaluations per step. The optimal order of convergence of any multipoint method without memory based on n evaluations is 2^{n-1}, [13]. Thus, the optimal order for a method with 3 function evaluations per step would be 4. The methods of Ostrowski [3], Jarratt [24], King [25] and Murakami [26] are optimal fourth-order methods. Nowadays, obtaining new optimal methods of order 4 is still important, because the corresponding efficiency index, 1.58740..., is very competitive and better than Cauchy's method $\sqrt[3]{3} = 1.442\ldots$ and Newton's method $\sqrt[2]{2} = 1.414\ldots$. Some papers with this aim have recently appeared in the literature (see [2,4,6,16,18]).

Cauchy's method, [21], is defined by (1) and (2) with

$$\phi(x) = \frac{2}{1+\sqrt{1-2L(x)}},$$

where

$$L(x) = \frac{f''(x)f(x)}{f'(x)^2}. \tag{4}$$

This is a well-known third-order method. There are numerous modifications of this method which are of the order four, for example, in the papers [14,16] its order is improved from three to four without the additional computational costs. However, in many cases, it is expensive to compute second derivatives and their practical applications are restricted rigorously, so that in past, Newton's method was frequently used to solve such non-linear equations because of its higher computational efficiency. Recently, many modified methods, free from the second derivative, have been studied in [15,19] and the literature cited therein. These methods have the order of convergence four, and therefore may be very interesting. The second derivative in (4) can be omitted by evaluating the function in different points.

Firstly, in this paper we consider simple modifications of Cauchy's method and we obtain three methods of fourth order. Two of these methods are free of the second derivative. Method from Kou [14], is also a special case of our method (8) with $z = \lambda$, defined by (6).

Using Padé approximation of order (k,m) to function $2/(1+\sqrt{1-2z})$ at 0, we obtain an infinite family of iterative methods. To this family belong the Newton's (order 2), Potra-Pták's [11], Chebyshev classical [21], Halley [27], Super-Halley [20], Newton–Steffensen, [22] (order 3) methods and some special cases of Murakami [26], Ostrowski [3], King [25], Jarratt [24], Chun et al.'s [4], Modified super-Halley, [17], Kou et al.'s, [19] (order 4). All methods of our family, except three, are optimal and of order four.

Under the assumptions, which are similar to these for the Newton's method in [8] global monotone convergence of third order for a subfamily of our new family of methods is proved. This subfamily is defined as $\varphi_{0,0},\ \varphi_{1,1},\ \varphi_{1,2},\ \varphi_{2,2},\ \varphi_{2,3}\ \varphi_{3,3},\ldots$, where function $\varphi_{k,m}$ is defined as Padé approximation of order (k,m) to the function $2/(1+\sqrt{1-2z})$ at 0. Some well-known methods belong to our subfamily, for example the Halley's method, the method (24) from [23] and the Super-Halley method from [20].

2. Main results

The Cauchy's method has the order of convergence three. It can be raised to four although the computational cost is not added if $L(x)$, (4), is replaced with one of the following functions

$$\sigma(x) = 2\frac{f(x-u(x))}{f(x)}, \tag{5}$$

$$\lambda(x) = \frac{3}{2}\left(1 - \frac{f'\left(x-\frac{2}{3}u(x)\right)}{f'(x)}\right), \tag{6}$$

$$\mu(x) = \frac{f(x)}{f'(x)^2} f'\left(x - \frac{1}{3}u(x)\right) \tag{7}$$

Theorem 1. *Assume that the function $f:(a,b)\subset R\to R$ has a simple root $\alpha\in(a,b)$. Let f be sufficiently smooth in the neighborhood of the root α, then the order of convergence of the modification of Cauchy's method*

$$x_{n+1} = x_n - \frac{2}{1+\sqrt{1-2z(x_n)}}u(x_n) \tag{8}$$