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Eigenvalues of a general class of boundary value problem with derivative-dependent nonlinearity

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ABSTRACT

We consider a general class of boundary value problem (BVP) comprising the differential equation

$$y^{(m)}(t) = \lambda F\big(t, y(t), y'(t), y''(t), \ldots, y^{(q)}(t)\big), \quad t \in (0,1)$$

where $1 \leqslant q \leqslant m-1$ and $\lambda > 0$, together with multi-point boundary conditions

$$y(0) = y'(0) = y''(0) = \cdots = y^{(q-1)}(0) = 0,$$
$$A_i\big(y^{(q)}(t_j), y^{(q+1)}(t_j), \ldots, y^{(m-1)}(t_j);\ 0 \leqslant j \leqslant r\big) = 0, \quad 1 \leqslant i \leqslant m-q$$

where $0 = t_0 < t_1 < \cdots < t_r = 1$. Particular cases of this general BVP include the well known Abel–Gontscharoff, focal, (m,p), Sturm–Liouville and complementary Lidstone BVPs. It is noted that BVPs with derivative-dependent nonlinear terms are less investigated in the literature due to technical difficulty. In this paper, a *new* technique is developed to characterize the eigenvalues λ so that the BVP has a positive solution. Explicit eigenvalue intervals are also established. We include several well known examples in the literature to illustrate the results obtained.

1. Introduction

In this paper we shall consider a general class of boundary value problem (BVP) comprising the differential equation

$$y^{(m)}(t) = \lambda F\big(t, y(t), y'(t), y''(t), \ldots, y^{(q)}(t)\big), \quad t \in (0,1) \tag{1.1}$$

where $m \geqslant 2,\ 1 \leqslant q \leqslant m-1,\ \lambda > 0$ and F is continuous at least in the interior of the domain of interest; together with q boundary conditions at $t = 0$

$$y(0) = y'(0) = y''(0) = \cdots = y^{(q-1)}(0) = 0 \tag{1.2}$$

and another $(m-q)$ boundary conditions involving $y^{(k)}(t_j)$'s where $k \in \{q, q+1, \ldots, m-1\}, j \in \{0, 1, \ldots, r\}$ and $0 = t_0 < t_1 < \cdots < t_r = 1$, these $(m-q)$ boundary conditions are generally represented by

$$A_i\big(y^{(q)}(t_j), y^{(q+1)}(t_j), \ldots, y^{(m-1)}(t_j); 0 \leqslant j \leqslant r\big) = 0, \quad 1 \leqslant i \leqslant m-q. \tag{1.3}$$

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The functions A_i are such that the Green's function exists for the BVP

$$\begin{cases} x^{(m-q)}(t) = 0, \quad t \in (0,1) \\ A_i\big(x(t_j), x'(t_j), \ldots, x^{(m-q-1)}(t_j); 0 \leqslant j \leqslant r\big) = 0, \quad 1 \leqslant i \leqslant m-q. \end{cases}$$

The BVP (1.1)–(1.3) is rather *general* and boundary conditions of type (1.2) and (1.3) include special cases such as Abel–Gontscharoff, focal, (m,p), Sturm–Liouville and complementary Lidstone, which are well known in the literature [2,7,4,9]. It is further noted that the nonlinear term F involves *derivatives* of the dependent variable, while most research papers on BVPs tackle nonlinear terms that involve y only.

We are interested in the existence of a positive solution of (1.1)–(1.3). By a *positive solution* y of (1.1)–(1.3), we mean a nontrivial $y \in C^{(m)}(0,1)$ satisfying (1.1) and $y(t) \geqslant 0$ for $t \in (0,1)$. If, for a particular λ the BVP (1.1)–(1.3) has a positive solution y, then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of (1.1)–(1.3). We shall denote the set of eigenvalues by E, i.e.,

$$E = \{\lambda > 0 |\ (1.1){-}(1.3) \text{ has a positive solution}\}.$$

The focus of this paper is the characterization of E. To be specific, we shall establish criteria for E to contain an interval, and for E to be an interval (which may either be bounded or unbounded). In addition explicit subintervals of E are derived.

There is a vast amount of research on the existence of positive solutions of BVPs for ordinary differential equations. For details of recent development in the field, the reader is referred to the monographs [2,4,6] and the hundreds of references cited therein. A substantial amount of the previous work focused on second order and fourth order problems as they have a wide range of applications, for example fourth order differential equations are used to model the deflection of elastic beams [31,41,42]. There are also many papers that deal with eigenvalue problems of BVPs with nonlocal boundary conditions, see for example [22,28,29,38,43]. It is noted that in most of these works the nonlinear terms considered do *not* involve derivatives of the dependent variable, only a handful of papers [16,17,24,34,37] tackle nonlinear terms that involve *even* order derivatives. We present below a brief survey of papers that have dealt with the existence of solutions of BVPs with nonlinear terms involving both even and odd derivatives:

(i) *Second order BVP*:

$$y''(t) = F(t, y(t), y'(t)), \quad t \in [0,1] \tag{1.4}$$

with three-point or four-point boundary conditions

$$\varphi(y(0), y'(0)) = 0, \quad \psi(y(1), y'(1)) = g(y(\eta)) \quad (0 < \eta < 1); \tag{1.5}$$

$$\varphi(y(0), y'(0)) = k(y(c)), \quad \psi(y(1), y'(1)) = g(y(d)) \quad (0 < c \leqslant d < 1) \tag{1.6}$$

have been discussed in [44,45] by the method of upper and lower solutions and Leray–Schauder degree. Here, F satisfies Nagumo type conditions and there are also monotonicity conditions imposed on the functions φ, ψ, k and g. For related work where F does not involve y', the reader may refer to [19,36,49,60].

(ii) *Fourth order BVP*:

$$y^{(4)}(t) = F\big(t, y(t), y'(t), y''(t), y^{(3)}(t)\big), \quad t \in [0,1] \tag{1.7}$$

with two-point boundary conditions

$$y(0) = y'(1) = a_0 y''(0) - b_0 y^{(3)}(0) = a_1 y''(1) - b_1 y^{(3)}(1) = 0 \tag{1.8}$$

has been studied in [47] using Leray–Schauder degree theory. Here, F satisfies Nagumo type condition and also F is monotone in certain arguments. Related papers on (1.7) and other boundary conditions include [12,18,20,39] where the method of upper and lower solutions is used and Nagumo type conditions are required for the nonlinear term F.

(iii) *Higher order BVP*:

$$y^{(n)}(t) = F\big(t, y(t), y'(t), \cdots, y^{(n-1)}(t)\big), \quad t \in [0,1] \tag{1.9}$$

subject to the following two-point boundary conditions

$$\begin{cases} y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-3 \\ a y^{(n-2)}(0) - b y^{(n-1)}(0) = A, \quad c y^{(n-2)}(1) + d y^{(n-1)}(1) = B \end{cases} \tag{1.10}$$

has been tackled in [23] via upper and lower solutions method, here F satisfies a Nagumo-type condition. A related problem can be found in [51] where F fulfills monotonicity type condition and the method of upper and lower solutions is again used. Further, (1.9) with some nonlinear two-point boundary conditions

$$\begin{cases} g_i\big(y^{(i)}(0), y^{(i+1)}(0), \cdots, y^{(n-1)}(0)\big) = 0, \quad 0 \leqslant i \leqslant n-2 \\ h\big(y(0), y'(0), \cdots, y^{(n-1)}(0); y(1), y'(1), \ldots, y^{(n-1)}(1)\big) = 0 \end{cases} \tag{1.11}$$