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Derivative free iterative methods for nonlinear systems

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A B S T R A C T

In this work we introduce a new operator of divided differences that preserves the convergence order when it is used for approximating the Jacobian matrix in iterative method for solving nonlinear systems. We obtain derivative free iterative methods with lower computational cost than the corresponding ones with different operators of divided differences. We also study the global convergence of these methods by analyzing their dynamical behavior.

1. Introduction

One of the most important problems in numerical analysis is approximating the solution of nonlinear equations using iterative methods. This fact has led to develop high order methods in order to solve different problems in mathematics and engineering, as can be seen in [1–4]. In some practical applications the derivative free methods are an important task in order to solve the problem, [5–8].

In a recent work, [9], we proved that adding a modified Newton's step to an iterative method of order p we obtain a method of order $p+2$. Our aim, now, is to extend this result to derivative free methods by showing that this result is preserved when we approximate the Jacobian matrix by a suitable operator of divided differences.

We prove that this technique can be applied repeatedly, so that new derivative free iterative methods with convergence order $p+2n$ can be obtained by applying it n times to a p order method.

Different studies in the scientific literature about the dynamic behavior of various iterative methods have been published, from the classical method of Newton, [10], to other higher order methods [11]. Recent papers show the dynamics of some derivative free methods, [12], applied to the solution of polynomial equations in the complex field.

Our aim is to study the dynamics of different derivative free iterative methods applied to systems of equations of real variables. Specifically, we show the basins attraction for a system of two polynomial equations of second degree in the real plane, representing the intersection of two conics.

The rest of the paper is organized as follows. A second order approximation of the Jacobian, called symmetric divided difference operator and its Taylor's expansion are obtained in Section 2. In Section 3, we prove theoretical results about the convergence order when we approximate the Jacobian matrix by symmetric divided differences. Section 4 examines different kinds of divided difference operators and Section 5 introduces new iterative methods obtained by substituting the derivatives by these operators. A complete study about the computational efficiency of the new methods is done in Section 6. In Section 7, the new methods are applied to some test equations. Finally, Section 8 examines the dynamics of some of these methods and Section 9 is devoted to the conclusions.

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2. Divided difference operators

Let be F a function $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ sufficiently differentiable in a convex set $D \subset \mathbb{R}^n$ containing α a simple zero of F.

In order to approximate the derivative, $F'(x)$, we consider the operator introduced in [13,14], (13, Appendix B) given by:

$$[x+h, x; F] = \int_0^1 F'(x+th)\mathrm{d}t, \quad (x,h) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1}$$

By integrating the Taylor's expansion of $F'(x+th)$ around x we have:

$$[x+h, x; F] = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3).$$

That is, it has been obtained an operator of divided differences that approximates the Jacobian $F'(x)$ with order h. A simple change of variable allows us to express the operator for the symmetric case:

$$[x+h, x-h; F] = \frac{1}{2}\int_{-1}^1 F'(x+th)\mathrm{d}t, \quad (x,h) \in \mathbb{R}^n \times \mathbb{R}^n,$$

then, integrating the same Taylor's development as before we obtain:

$$[x+h, x-h; F] = F'(x) + \frac{1}{6}F'''(x)h^2 + O(h^3), \tag{2}$$

that is, an approximation of second order for the Jacobian, that is called symmetric approximation.

3. Main result

In [15] we introduce a technique for increasing from p to $p+2$ the convergence order of an iterative method. The procedure consists in composing the iterative method of order p with a modification of Newton's method that introduces just a new functional evaluation avoiding the calculation of the new derivative.

In this work we show that this improvement in the convergence order holds if we replace all the derivatives by their symmetric divided differences approach. Then, new derivative free iterative methods for nonlinear systems are obtained.

The iterative expression of the new methods is as follows:

$$\begin{aligned} y_n &= x_n - [x_n+h, x_n-h; F]^{-1}F(x_n), \\ z_n &= \phi(x_n, y_n), \\ w_n &= z_n - [y_n+h, y_n-h; F]^{-1}F(z_n), \end{aligned} \tag{3}$$

where $[u+h, u-h; F]$ is a symmetric approximation by divided differences for $F'(u)$ with $h = F(u)$ and ϕ is the iteration function of an iterative method of convergence order $p \geqslant 2$. We establish the following result:

Theorem 1. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a sufficiently smooth function in a neighborhood of α, a simple zero of $F(x) = 0$, in $D \subset \mathbb{R}^n$ and the iterative method $z_n = \phi(x_n, y_n)$ with convergence order p. Then, method (3) has convergence order $p+2$.*

Proof. We consider the Taylor's expansion of $F(x_n)$ in α:

$$F(x_n) = \Gamma e_n + A_2\Gamma e_n^2 + A_3\Gamma e_n^3 + A_4\Gamma e_n^4 + A_5\Gamma e_n^5 + O(e_n^6),$$

being $\Gamma = F'(\alpha)$, $e_n = x_n - \alpha$ and $A_k = \frac{F'(\alpha)^{-1}F^{(k)}(\alpha)}{k!} \in \mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$, $k = 2, 3, \ldots$.

Then, the derivatives of $F(x_n)$ in a neighborhood of α take the form:

$$\begin{aligned} F'(x_n) &= \Gamma + 2A_2\Gamma e_n + 3A_3\Gamma e_n^2 + 4A_4\Gamma e_n^3 + 5A_5\Gamma e_n^4 + O(e_n^5), \\ F''(x_n) &= 2A_2\Gamma + 6A_3\Gamma e_n + 12A_4\Gamma e_n^2 + 20A_5\Gamma e_n^3 + O(e_n^4), \\ F'''(x_n) &= 6A_3\Gamma + 24A_4\Gamma e_n + 60A_5\Gamma e_n^2 + O(e_n^3). \end{aligned}$$

By substituting these expansions in (2) with $x = x_n$ and $h = F(x_n)$, we have the error equation for the symmetric divided difference approximation:

$$\begin{aligned} [x_n+h, x_n-h; F] = \Gamma &+ 2A_2\Gamma e_n + A_3\Gamma\left(\Gamma^2+3\right)e_n^2 + \left(2A_2A_3\Gamma^3 + 4A_4\left(\Gamma^2+1\right)\Gamma\right)e_n^3 \\ &+ \left(2A_3^2\Gamma^3 + A_2^2A_3\Gamma^3 + 8A_2A_4\Gamma^3 + 5A_5\left(2\Gamma^2+1\right)\Gamma\right)e_n^4 + O(e_n^5) \end{aligned}$$

and then we can deduce the error equation for its inverse operator, in the way it is done in [16] (see Eqs. (2) and (3)):