Applied Mathematics and Computation 259 (2015) 753–761

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On positive solutions for a second order differential system with indefinite weight ☆

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ARTICLE INFO

Keywords:
Positive solutions
Indefinite weight
Fixed point theorem

ABSTRACT

In this paper, we consider the existence of positive solutions for a second order differential system

$$\begin{cases} -u'' = a(t)\varphi u + h(t)f(u), & 0 < t < 1, \\ -\varphi'' = b(t)u, & 0 < t < 1, \\ u(0) = u(1) = 0, \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

where $a(t), b(t), h(t)$ change sign. The proof is based on the well-known fixed point theorem of cone expansion and compression.

1. Introduction

The purpose of this work is to study the existence of positive solutions for a second order differential system

$$\begin{cases} -u'' = a(t)\varphi u + h(t)f(u), & 0 < t < 1, \\ -\varphi'' = b(t)u, & 0 < t < 1, \\ u(0) = u(1) = 0, \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{1}$$

where $a(t), b(t), h(t)$ may change sign. By a positive solution, we mean that a pair of functions (u, φ) with $u, \varphi \in C^2(0,1) \cap C[0,1]$ is a positive solution of (1), if (u, φ) satisfies (1), and $u,\ \varphi \geqslant 0, t \in [0,1]$, $u,\ \varphi \not\equiv 0$. During the last few decades, similar problems have been widely investigated and it is well known they have a strong physical meaning because they appear in quantum mechanics models [1,2], in semiconductor theory [3] or a time and space-dependent mathematical model of nuclear reactors in a closed container [4]. On one hand, in [5], Gu and Wang prove that there is at least one positive stationary solution for the reaction–diffusion system

☆ This research is supported by Natural Science Foundation of JiangSu Province (Grant No. 1014-51314411).
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http://dx.doi.org/10.1016/j.amc.2015.02.089
0096-3003/

$$\begin{cases} u_{1t} - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ u_{2t} - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ u_1 = u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \geqslant 0,\ u_2(x,0) = u_{20}(x) \geqslant 0, & x \in \overline{\Omega}, \end{cases}$$

where $\Omega \in R^N (2 \leqslant N < 6)$ is a smooth bounded domain, $a, b > 0$ are constants, u_{10}, u_{20} are continuous nonnegative functions on $\overline{\Omega}$. In [6], Wang and An investigate the one-dimensional system such as:

$$\begin{cases} -u'' + \lambda u = \varphi u + f(t,u), & 0 < t < 1, \\ -\varphi'' = \mu u, & 0 < t < 1 \\ u(0) = u(1) = 0, \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{2}$$

where $\lambda > -\pi^2$. The proof is based on the fixed point theorem of cone expansion and compression. Lately, Chen and Ma show that (2) has at least one positive solution for $\mu \in (0, +\infty)$ using bifurcation techniques in [7]. The similar results are obtained for the positive radial solutions of (2) for higher dimension R^N in [8]. But in the above references [5–8], the nonlinearity is required to be nonnegative.

On the other hand, since the excellent results obtained by Cac etc in [9], many authors have studied the existence and multiplicity of positive solutions for differential equations with nonlinearity that changes sign, such as: In [10,11], the authors study existence, multiplicity and stability of positive solutions of an indefinite weight boundary value problem

$$\begin{cases} -u'' = \lambda a(t) f(u), & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

by the fixed point or bifurcation techniques. In [12,13], the authors obtain the existence of positive solutions for the elliptic system or telegraph system using the method of monotone iteration and Schauder fixed point theorem.

Therefore, inspired by these references, we will study the existence of positive solutions for (1), if the function $a(t), b(t)$, $h(t)$ satisfy the changing sign condition:(H1) $a, b, h : [0,1] \to (-\infty, +\infty)$ are continuous, and there exists a constant $\xi \in (0,1)$ such that

$$\begin{cases} a(t),\ b(t),\ h(t) \geqslant 0, & \text{if } t \in [0,\xi], \\ a(t),\ b(t),\ h(t) \leqslant 0 & \text{if } t \in [\xi,1]. \end{cases}$$

Moreover, $a(t), b(t), h(t)$ do not vanish identically on any subinterval of [0, 1].

The main proof is based on the well-known fixed point theorem of cone expansion and compression [14].

Lemma 1.1. *Let E be a Banach space, and $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \leqslant \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \geqslant \|u\|,\ u \in K \cap \partial\Omega_2$; or
(ii) $\|Tu\| \geqslant \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \leqslant \|u\|,\ u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main result

Theorem 2.1. *Assume that (H1) holds. In addition, the following conditions hold:*

(H2) There exists $0 < \sigma_1 < \xi$ such that

$$\sigma_1 \int_{\sigma_1}^{\xi} G(t,s) b^+(s) ds \geqslant \xi \int_{\xi}^{1} G(t,s) b^-(s) ds;$$

(H3) There exists $0 < \sigma_2 < \xi$ such that

$$\sigma_2 \int_{\sigma_2}^{\xi} G(t,s) G(s,s) a^+(s) ds \geqslant \xi \int_{\xi}^{1} G(t,s) a^-(s) ds,$$

where $G(t,s)$ is the Green function of linear boundary value problem

$$-u'' = 0, \quad u(0) = u(1) = 0,$$