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Metamathematical investigations on the theory of Grossone

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ABSTRACT

We propose an axiomatization of Sergeyev's theory of Grossone, trying to comply with his methodological principles. We find that a simplified form of his Divisibility axiom is sufficient. We use for easier readability a second order language and a predicative second order logic. Our theory is not finitely axiomatizable and is a conservative extension of Peano's arithmetic.

1. Introduction

In the last ten years Yaroslav D. Sergeyev has introduced a new methodology for computing with infinities and infinitesimals, and he has applied it to a variety of problems, from numerical computations to ODE, Riemann's Zeta function, fractals and Turing machines.[1] Sergeyev's treatment is based on the use of a symbol ① for a numeral, called *Grossone*, which is meant to denote the number of elements of the set $\mathbb{N}$ of natural numbers. His theory differs both from the classical Cantorian introduction of infinite numbers and from non-standard analysis, as well as from Benci and Di Nasso's numerosity investigations in [9]. A first appraisal of Sergeyev's proposal against the background of the contemporary renewed interest in infinitesimals has been sketched by the author in [10].

The new numeral system conceived by Sergeyev aims at obtaining more accurate results concerning the infinite. It allows e.g. to assign different numbers to the set of natural numbers and to the set of even numbers, and quite properly the first number turns out to be the double of the latter.

Sergeyev's approach is more similar to that of an empirical scientist than of a deductive mathematician. He credits to himself unassumingly only the introduction of "a new computational methodology", which he justifies with the metaphor of the instruments resolution power: "Physicists decide the level of the precision they need and obtain a result depending of the chosen level of the accuracy. In the moment when they put a lens in the microscope, they have decided the minimal (and the maximal) measure of objects that they will be able to observe. If they need a more precise or a more rough answer, they change the lens of their microscope [$\cdots$] In natural sciences there always exists the triad – the researcher, the object of investigation, and tools used to observe the object – and the instrument used to observe the object bounds and influences results of observations. The same happens in Mathematics studying numbers and objects that can be constructed by using numbers. Numeral systems used to express numbers are instruments of observations used by mathematicians" [5, p. 131].

Sergeyev is wary of the axiomatic method because he thinks that by adopting it we would be tied to the expressive power of a language in the description of mathematical objects and concepts (see [3, Section 2]); on the contrary, "the choice of the mathematical language depends on the practical problem that is to be solved and on the accuracy required for such a

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[1] See e.g. Sergeyev's [1–5], Sergeyev and Garro's [6,7] and De Cosmis and De Leone's [8] for applications to Linear Programming.

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solution. In dependence of this accuracy, a numeral system that would be able to express the numbers composing the answer should be chosen" [5, p. 131].

Actually no mathematician ever works in a fixed theory. This is why logic is not popular among them. However, to assess the strength of an instrument, we have to consider it so to say *in vitro*.

In this paper we investigate the possibility of a formal axiomatic presentation of the arithmetical theory of ①,[2] at the risk of being unfaithful to Sergeyev's spirit; our aim is to make the theory accessible and acceptable, hopefully palatable, to traditionally minded mathematicians and to discuss a few of its metamathematical properties. To be able to offer a clear axiomatization, we chose to use a predicative second order logic, instead of first order logic, for reasons that will become clear in the ensuing text. Details will be provided for those who are not familiar with logical matters.

2. An informal presentation of Grossone

Sergeyev's philosophy is expressed by three postulates he assumes before plunging ahead into the computational methodology:

P1. We postulate the existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only a finite number of operations.

P2. We shall not tell what are the mathematical objects we deal with. Instead, we shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.

P3. We adopt the principle: 'The part is less than the whole', and apply it to all numbers, be they finite, infinite, or infinitesimal, as well as to all sets and processes, be they finite or infinite" [3, Section 2].

These three postulates constitute an unusual introduction to a mathematical theory. Their meaning becomes clearer with the development of the mathematical content, but a few preliminary comments are not out of place.

The postulates are not to be conceived as the "axioms" of an axiomatic theory. They have a pragmatic or methodological character, P2 and P3 a more general, P1 a more binding one; they explain how users of the methodology work and produce their results.

The principle in P3 goes contrary to the ideas prevailing after the success of set theory; its statement prepares the readers to accept results which are at odds with the received wisdom and it probably reveals what was behind the original intuition of the new conception. P3 is actually Euclid's "common notion" n. 5 (see [11, vol. 1, p. 155]).

All three postulates could probably be likened to Euclid's common notions, at least to n. 5, not to the others, which were basically properties of equality. According to Proclus, Euclid's common notions were general axioms not specific to geometry but valid for all sciences, as contemplated by Aristotle's theory of science.[3]

P1 and P2 are the programmatic statements of the new methodology, and have a multiple function. P1 explains a constraint to be obeyed that affects the results that will be obtained. Probably it is also a warning that if not respected a contradiction could follow. So it is a restriction of the acceptable arguments.

P2 is a reference to the philosophy of the instruments accuracy mentioned earlier. Mathematicians normally do not say what their objects are, so its first sentence could be superfluous; but mathematicians usually tacitly assume that in case of need their objects could be defined, for example in set theory (unless they are full fledged formalists); no such eventuality can present itself in this case, since the focus is on tools. It would be useless to ask Sergeyev what are the objects he studies with his new tools, and if he is a realist or otherwise; very likely he would not take the bait, insisting that objects being what representation systems allow us to discover, we can only say of them what the instruments we use allow us to say.

Sergeyev's position does not fall under any of the traditional philosophies of mathematics: he accepts the realistic talk of the working mathematician but qualifies it as we have seen above; he has a finitistic restriction on logic, but he is not a constructivist. One could possibly draw a parallel with P.W. Bridgman's operationalism for the physical sciences, see [12], but the issue deserves a deeper scrutiny.

After the postulates, the first step is the introduction of the so called "Infinite Unit Axiom" (IUA), which consists of three parts, namely (from [3, Section 3]):"*Infinity*. Any finite natural number n is less than Grossone, i.e., $n < ①$.

Identity. The following relations link ① to identity elements 0 and 1

$$0 \cdot ① = ① \cdot 0 = 0,\ \ ① - ① = 0, ①/① = 1,\ \ ①^0 = 1,\ \ 1^① = 1,\ \ 0^① = 0.$$

Divisibility. For every finite natural number k, sets $\mathbb{N}_{k;n}, 1 \leqslant k \leqslant n$, being the n-th part of the set, $\mathbb{N}$, of natural numbers have the same number of elements indicated by the numeral $①/n$ where.

[2] Grossone as a natural number is also of course a real number, and as such treated in many of Sergeyev's investigations. But we consider it here only in the arithmetical context.

[3] The fifth common notion does not seem to be genuine however, being a generalization of a geometric inference used only in Euclid I,6, as noted in [11, vol. 1, p. 232].