Applied Mathematics and Computation 255 (2015) 36–43

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Grossone approach to Hutton and Euler transforms

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ARTICLE INFO

Keywords:
Divergent series
Summability
Hyperfinite domain

ABSTRACT

The aim of this paper is to demonstrate that several non–rigorous methods of mathematical reasoning in the field of divergent series, mostly related to the Euler and Hutton transforms, may be developed in a correct and consistent way by methods of the grossone analysis.

1. Introduction

We begin with the following sequence of symbolic transformations with the shift operator, related to summability of divergent series.

Shift operator. An arbitrary series

$$A = a_0 + a_1 + a_2 + a_3 + \cdots \tag{1}$$

can be rewritten as

$$A = a_0 + \tau a_0 + \tau^2 a_0 + \tau^3 a_0 + \cdots = (1 + \tau + \tau^2 + \tau^3 + \cdots)\, a_0, \tag{2}$$

where τ is the *shift operator*, an operator of yet indefinite mathematical nature, but acting so that $\tau a_k = a_{k+1}$, — and then as

$$A \;=\; \frac{1}{1-\tau}\, a_0, \tag{3}$$

summing up $1 + \tau + \tau^2 + \tau^3 + \cdots$ according to the informal equality

$$1 + \tau + \tau^2 + \tau^3 + \cdots \;=\; \frac{1}{1-\tau}. \tag{4}$$

Hutton transform. Now, let $d \neq -1$ and $\sigma[d] = d + \tau$. Formally,

$$\left.\begin{aligned} \tfrac{1}{1-\tau} &= \tfrac{1-\tau+d+\tau}{(1+d)(1-\tau)} = \tfrac{1}{1+d} + \tfrac{1}{1-\tau}\,\tfrac{\sigma[d]}{1+d} && = \\ &= \tfrac{1}{1+d} + (1 + \tau + \tau^2 + \cdots)\,\tfrac{\sigma[d]}{1+d} && = \\ &= \tfrac{1}{1+d} + \tfrac{d+\tau}{1+d} + \tfrac{d\tau+\tau^2}{1+d} + \tfrac{d\tau^2+\tau^3}{1+d} + \cdots && \end{aligned}\right\} \tag{5}$$

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1 Partial financial support of Grant RFBR 13-01-00006 acknowledged.

http://dx.doi.org/10.1016/j.amc.2014.06.037
0096-3003/

and we get *the Hutton transform* (H, d) (Hardy [1] for $d = 1$) of the original series,

$$A = \frac{a_0}{1+d} + \frac{da_0 + a_1}{1+d} + \frac{da_1 + a_2}{1+d} + \frac{da_2 + a_3}{1+d} + \cdots. \tag{6}$$

Iterated Hutton, or Euler–Jakimovski transform. Let $\{d_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers $d_n \neq -1$. Applying transformations $(H, d_1),\ (H, d_2),\ (H, d_3), \ldots$ – as in (5) and (6) – consecutively, so that the first term of every intermediate series is separated after each iteration, we obtain the series of separated (frameboxed) terms as the final result:

$$\left.\begin{aligned}
\tfrac{1}{1-\tau} &= \boxed{\tfrac{1}{1+d_1}} + \left(\tfrac{d_1+\tau}{1+d_1}\,\tfrac{1}{1-\tau}\right) = \\
&= \tfrac{1}{1+d_1} + \tfrac{d_1+\tau}{1+d_1}\left(\tfrac{1}{1+d_2} + \tfrac{d_2+\tau}{1+d_2}\,\tfrac{1}{1-\tau}\right) = \\
&= \boxed{\tfrac{1}{1+d_1} + \tfrac{d_1+\tau}{(1+d_1)(1+d_2)}} + \left(\tfrac{(d_1+\tau)(d_2+\tau)}{(1+d_1)(1+d_2)}\,\tfrac{1}{1-\tau}\right) = \\
&= \boxed{\tfrac{1}{1+d_1} + \tfrac{d_1+\tau}{(1+d_1)(1+d_2)} + \tfrac{(d_1+\tau)(d_2+\tau)}{(1+d_1)(1+d_2)(1+d_3)}} + \\
&\qquad + \left(\tfrac{(d_1+\tau)(d_2+\tau)(d_3+\tau)}{(1+d_1)(1+d_2)(1+d_3)}\,\tfrac{1}{1-\tau}\right) = \\
&\ldots \qquad \ldots \qquad \ldots \\
&= \sum_{k=0}^{\infty} \tfrac{(d_1+\tau)(d_2+\tau)\ldots(d_k+\tau)}{(1+d_1)(1+d_2)\ldots(1+d_k)(1+d_{k+1})}.
\end{aligned}\right\} \tag{7}$$

Remark 1. The final series is a formal Newton's interpolation of the function $\frac{1}{1-\tau}$ with the nodes $-d_1, -d_2, -d_3, \ldots$.

We conclude that, in the spirit of (3),

$$A = \frac{1}{1-\tau}\, a_0 \;=\; \sum_{k=0}^{\infty} \frac{(d_1+\tau)(d_2+\tau)\ldots(d_k+\tau)}{(1+d_1)(1+d_2)\ldots(1+d_k)(1+d_{k+1})}\, a_0, \tag{8}$$

where each polynomial $P_k(\tau) = (d_1+\tau)(d_2+\tau)\ldots(d_k+\tau)$ formally acts on a_0 in accordance with the basic equalities $\tau^n a_0 = a_n$.

Remark 2. Transformation (7) and (8) was explicitly introduced by Jakimovski [2] (as $[F, d_n]$) based on a series of earlier studies. Yet most notably, the whole idea of iterated transformation with separation of first terms of intermediate series belongs to Leonhard Euler, *Institutiones Calculi Differentialis*, Part II, Section 10 – see a discussion in Hardy [1, Section 2.6]. This is why we call it the *Euler–Jakimovski transformation* here. The summability method based on the Euler–Jakimovski transformation works, pending appropriate choice of d_n, for rapidly divergent oscillating series like $0! - 1! + 2! - 3! + \cdots$. See [3] for further references. □

2. Regression: some linear transformations

The transformations considered above can be represented by the following infinite matrices:

$$H(d) = \frac{1}{1+d}\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ d & 1 & 0 & 0 & \ldots \\ 0 & d & 1 & 0 & \ldots \\ 0 & 0 & d & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \quad S(d) = \begin{pmatrix} d & 1 & 0 & 0 & \ldots \\ 0 & d & 1 & 0 & \ldots \\ 0 & 0 & d & 1 & \ldots \\ 0 & 0 & 0 & d & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$E(\{d_n\}_{n=1}^{\infty}) = \begin{pmatrix} \frac{1}{D_0} & 0 & 0 & 0 & 0 & \ldots \\ \frac{d_1}{D_1} & \frac{1}{D_1} & 0 & 0 & 0 & \ldots \\ \frac{d_1 d_2}{D_2} & \frac{d_1+d_2}{D_2} & \frac{1}{D_2} & 0 & 0 & \ldots \\ \frac{d_1 d_2 d_3}{D_3} & \frac{d_1 d_2 + d_1 d_3 + d_2 d_3}{D_3} & \frac{d_1+d_2+d_3}{D_3} & \frac{1}{D_3} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

where $D_k = (1+d_1)(1+d_2)\ldots(1+d_{k+1})$ – so that

$$[E]_k \;=\; \frac{1}{1+d_{k+1}}\left[\frac{S(d_1)}{1+d_1}\cdot\frac{S(d_2)}{1+d_2}\cdot\ldots\cdot\frac{S(d_k)}{1+d_k}\right]_0, \qquad k = 0, 1, 2, \ldots, \tag{9}$$

where $[M]_k,\ k = 0, 1, 2, \ldots$, is the kth row of any matrix M.