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Analysis on the Levi-Civita field and computational applications

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This paper is dedicated to the loving memory of my brother Saïd Shamseddine (1968–2013).

ABSTRACT

In this paper, we present an overview of some of our research on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$. $\mathcal{R}$ (resp. $\mathcal{C}$) is the smallest non-Archimedean field extension of the real (resp. complex) numbers that is Cauchy-complete and real closed (resp. algebraically closed); in fact, $\mathcal{R}$ is small enough to allow for the calculus on the field to be implemented on a computer and used in applications such as the fast and accurate computation of the derivatives of real functions as “differential quotients” up to very high orders. We summarize the convergence and analytical properties of power series, showing that they have the same smoothness behavior as real and complex power series; we present a Lebesgue-like measure and integration theory on the Levi-Civita field $\mathcal{R}$; we discuss solutions to one-dimensional and multi-dimensional optimization problems based on continuity and differentiability concepts that are stronger than the topological ones; and we give a brief summary of the results of our ongoing work on developing a non-Archimedean operator theory on a Banach space over $\mathcal{C}$.

1. Introduction

An overview of recent research on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ will be presented. We recall that the elements of $\mathcal{R}$ and its complex counterpart $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1 ($\lambda, \sim, \approx$). For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(\text{supp}(x))$, which exists because of the left-finiteness of $\text{supp}(x)$; and we let $\lambda(0) = +\infty$. Moreover, we denote the value of x at $q \in \mathbb{Q}$ with brackets like $x[q]$.

Given $x,\ y \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [5] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R},\ \mathbb{C} \to \mathcal{R},\ \mathcal{C}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases}. \tag{1.1}$$

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Definition 1.2 (*Order in* $\mathcal{R}$). Let $x, y \in \mathcal{R}$ be given. Then we say that $x > y$ (or $y < x$) if $x \neq y$ and $(x-y)[\lambda(x-y)] > 0$; and we say $x \geqslant y$ (or $y \leqslant x$) if $x = y$ or $x > y$.

It follows that the relation $\geqslant$ (or $\leqslant$) defines a total order on $\mathcal{R}$ which makes it into an ordered field. Note that, given $a < b$ in $\mathcal{R}$, we define the $\mathcal{R}$-interval $[a,b] = \{x \in \mathcal{R} : a \leqslant x \leqslant b\}$, with the obvious adjustments in the definitions of the intervals $[a,b[,]a,b]$, and $]a,b[$. Moreover, the embedding Π in Eq. (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order.

The order leads to the definition of an ordinary absolute value on $\mathcal{R}$:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0; \end{cases}$$

which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology) as that induced by the ultrametric absolute value:

$$|x|_u = e^{-\lambda(x)},$$

as was shown in [36]. Moreover, two corresponding absolute values are defined on $\mathcal{C}$ in the natural way:

$$|x+iy| = \sqrt{x^2+y^2}; \text{and} |x+iy|_u = e^{-\lambda(x+iy)} = \max\{|x|_u, |y|_u\}.$$

Thus, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ (or $|\cdot|_u$) in $\mathcal{R}$.

We note in passing here that $|\cdot|_u$ is a non-Archimedean valuation on $\mathcal{R}$ (resp. $\mathcal{C}$); that is, it satisfies the following properties

(1) $|v|_u \geqslant 0$ for all $v \in \mathcal{R}$ (resp. $v \in \mathcal{C}$) and $|v|_u = 0$ if and only if $v = 0$;
(2) $|vw|_u = |v|_u|w|_u$ for all $v, w \in \mathcal{R}$ (resp. $v, w \in \mathcal{C}$); and
(3) $|v+w|_u \leqslant \max\{|v|_u, |w|_u\}$ for all $v, w \in \mathcal{R}$ (resp. $v, w \in \mathcal{C}$): the strong triangle inequality.

Thus, $(\mathcal{R}, |\cdot|)$ and $(\mathcal{C}, |\cdot|)$ are non-Archimedean valued fields.

Besides the usual order relations on $\mathcal{R}$, some other notations are convenient.

Definition 1.3 ($\ll, \gg$). Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4 (*The Number d*). Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $\text{supp}(x)$ can be arranged in ascending order, say $\text{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the valuation topology [5].

Altogether, it follows that $\mathcal{R}$ (resp. $\mathcal{C}$) is a non-Archimedean field extension of $\mathbb{R}$ (resp. $\mathbb{C}$). For a detailed study of these fields, we refer the reader to [5,32,26,6,33,34,40,7,35,41,36,37,28,38,39,30,1,31]. In particular, it is shown that $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the natural (valuation) topology.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [20]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [19], and for an overview of the related valuation theory to the books by Krull [11], Schikhof [20] and Alling [3]. A thorough and complete treatment of ordered structures can also be found in [18].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [5]; and having infinitely small numbers in the field allows for many computational applications similar to those obtained with the numerical system employed by Sergeyev in [21–25]. One such application is the computation of derivatives of real functions representable on a computer [32], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

In the following sections, we present a brief overview of recent research done on $\mathcal{R}$ and $\mathcal{C}$; and we refer the interested reader to the respective papers for a more detailed study of any of the research topics summarized below.

2. Calculus on $\mathcal{R}$

The following examples show that functions on a finite interval of $\mathcal{R}$ behave in a way that is different from (and even opposite to) what we would expect under similar conditions in $\mathbb{R}$.