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Global solutions to fractional programming problem with ratio of nonconvex functions

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ABSTRACT

This paper presents a canonical dual approach for minimizing a sum of quadratic function and a ratio of nonconvex functions in $\mathbb{R}^n$. By introducing a parameter, the problem is first equivalently reformed as a nonconvex polynomial minimization with elliptic constraint. It is proved that under certain conditions, the canonical dual is a concave maximization problem in $\mathbb{R}^2$ that exhibits no duality gap. Therefore, the global optimal solution of the primal problem can be obtained by solving the canonical dual problem.

1. Introduction

We intend to solve the following nonconvex fractional programming problem:

$$(\mathcal{P}): \quad \min\left\{P_0(\mathbf{x}) = f(\mathbf{x}) + \frac{g(\mathbf{x})}{h(\mathbf{x})} : \quad \mathbf{x} \in \mathcal{X}\right\}, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ and

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x},$$

$$g(\mathbf{x}) = \frac{1}{2}\left(\frac{1}{2}|B\mathbf{x}|^2 - \lambda\right)^2 - \mathbf{c}^T\mathbf{x},$$

$$h(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x},$$

with $B \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{n\times n}$ being symmetric, $H \in \mathbb{R}^{n\times n}$ negative definite, $\lambda \in \mathbb{R}^+$, and $\mathbf{f},\ \mathbf{c},\ \mathbf{b} \in \mathbb{R}^n$ are given vectors. In this paper, the notation $|v|$ denotes the Euclidean norm of v. Assume that $\mu_0^{-1} = h(H^{-1}\mathbf{b}) > 0$ and $\delta \in (0, \mu_0^{-1}]$, the feasible domain $\mathcal{X}$ is defined by

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geqslant \delta > 0\},$$

which is a constraint of elliptic type.

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Problem $(\mathcal{P})$ belongs to a class of "sum-of-ratios" problems that have been actively studied for several decades. The ratios often stand for efficiency measures representing performance-to-cost, profit-to-revenue or return-to-risk for numerous applications in economics, transportation science, finance, and engineering (see [1–4,6,8–12,22–27,29,31–33]). The nonconvex function $g(\mathbf{x})$ is the well-known *double-well potential*, which appears frequently in chaotic dynamics [15], phase transitions of solids [16], and large deformation mechanics [21]. Depending on the nature of each application, the functions f, g, h can be affine, convex, concave, or neither. However, even for the simplest case in which f, g, h are all affine functions, the problem $(\mathcal{P})$ is still a global optimization problem that may have multiple local optima [5,28]. In particular, Freund and Jarre [13] showed that the sum-of-ratios problem $(\mathcal{P})$ is NP-complete when f, g are convex and h is concave.

Canonical duality theory is a powerful methodological theory which can be used for solving a large class of nonconvex/nonsmooth/discrete problems in nonlinear analysis and global optimization [14,18,19]. The main goal of this paper is to solve the problem $(\mathcal{P})$ by the canonical duality theory. In Section 2, the problem $(\mathcal{P})$ is first parameterized equivalently as a nonconvex polynomial minimization $(\mathcal{P}_\mu)$ with an elliptic constraint. For each given parameter, the canonical dual problem is derived by the standard canonical dual transformation. The global optimality condition is proposed in Section 3. An example is illustrated in Section 4. Conclusion is provided in the last section.

2. Canonical dual problem

In order to solve the problem $(\mathcal{P})$, we consider the following parameterized subproblem:

$$(\mathcal{P}_\mu): \quad \min\left\{P_\mu(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x}) \ : \quad \mathbf{x} \in \mathcal{X}_\mu\right\}, \tag{2}$$

where $\mu \in [\mu_0, \delta^{-1}]$ and

$$\mathcal{X}_\mu = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geqslant \mu^{-1} \geqslant \delta > 0\}$$

is a convex set. We immediately have the following result:

Lemma 1. *Problem $(\mathcal{P})$ is equivalent to $(\mathcal{P}_\mu)$ in the sense that*

$$\inf_{\mathbf{x}\in\mathcal{X}} P_0(\mathbf{x}) = \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{\mathbf{x}\in\mathcal{X}_\mu} P_\mu(\mathbf{x}). \tag{3}$$

Proof. It is easy to see that

$$\inf_{\mathbf{x}\in\mathcal{X}} P_0(\mathbf{x}) = \inf_{\mathbf{x}\in\mathcal{X}}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \frac{g(\mathbf{x})}{h(\mathbf{x})}\right\} = \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{h(\mathbf{x})=\mu^{-1}}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \frac{g(\mathbf{x})}{h(\mathbf{x})}\right\}$$
$$= \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{h(\mathbf{x})=\mu^{-1}}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x})\right\} \geqslant \mathit{inf}_{\mu\in[\mu_0,\delta^{-1}]} \inf_{\mathbf{x}\in\mathcal{X}_\mu}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x})\right\} = \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{\mathbf{x}\in\mathcal{X}_\mu} P_\mu(\mathbf{x}).$$

Conversely,

$$\inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{\mathbf{x}\in\mathcal{X}_\mu}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x})\right\} = \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{h(\mathbf{x})\geqslant\mu^{-1}}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x})\right\}$$
$$\geqslant \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{h(\mathbf{x})\geqslant\mu^{-1}}\left\{\frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \frac{g(\mathbf{x})}{h(\mathbf{x})}\right\} \text{ (since } g(\mathbf{x}) > 0) = \inf_{\mathbf{x}\in\mathcal{X}} P_0(\mathbf{x}).$$

This completes the proof of the lemma. □

Now, for any $\mu \in [\mu_0, \delta^{-1}]$, we define

$$G_\mu(\varsigma, \sigma) = Q + \mu\varsigma B^T B - \sigma H, \tag{4}$$

$$\mathcal{S}_\mu^+ = \{(\varsigma, \sigma) \in \mathbb{R}^2 \mid \varsigma \geqslant -\lambda, \quad \sigma \geqslant 0, \quad G_\mu(\varsigma, \sigma) \succ 0\}, \tag{5}$$

where '$\succ$' means positive definiteness of a matrix. Let $\partial\mathcal{S}_\mu^+$ denote a singular hyper-surface defined by

$$\partial\mathcal{S}_\mu^+ = \{\varsigma \geqslant -\lambda, \sigma \geqslant 0 \mid G_\mu(\varsigma, \sigma) \succeq 0, \ \det G_\mu(\varsigma, \sigma) = 0\}. \tag{6}$$

Then, the parametrical canonical dual problem can be proposed as the following:

$$P_\mu^d(\varsigma, \sigma) = -\frac{1}{2}(\mathbf{f} + \mu\mathbf{c} - \sigma\mathbf{b})^T G_\mu^{-1}(\varsigma, \sigma)(\mathbf{f} + \mu\mathbf{c} - \sigma\mathbf{b}) - \mu\lambda\varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu}. \tag{7}$$

Therefore, the canonical dual problem $(\mathcal{P}_\mu^d)$ can be proposed as the following: