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On the numerical solution of some nonlinear and nonlocal boundary value problems

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A B S T R A C T

The modeling of various physical questions often leads to nonlinear boundary value problems involving a nonlocal operator, which depends on the unknown function in the entire domain, rather than at a single point. In order to answer an open question posed by J.R. Cannon and D.J. Galiffa, we study the numerical solution of a special class of nonlocal nonlinear boundary value problems, which involve the integral of the unknown solution over the integration domain. Starting from Cannon and Galiffa's results, we provide other sufficient conditions for the unique solvability and a more general convergence theorem. Moreover, we suggest different iterative procedures to handle the nonlocal nonlinearity of the discrete problem and show their performances by some numerical tests.

1. Introduction

We consider the problem of finding a numerical solution of the following homogeneous, nonlocal, nonlinear boundary value problem

$$-\alpha\left(\int_0^1 u(t)dt\right)u''(x) + [u(x)]^{2n+1} = 0, \quad x \in (0,1),\ u(0) = a,\ u(1) = b, \tag{1.1}$$

where $\alpha(q)$ is a given positive function of $q > 0$, n is a positive integer, a, b are positive real constants and u is the sought solution.

This problem falls within the class of "nonlocal nonlinear" problems because its nonlinearity is not only local (we refer to the presence of $[u(x)]^{2n+1}$), but it is also nonlocal since the coefficient of the derivative of the unknown solution u depends upon the integral of u itself, which in turn depends on the whole domain $(0,1)$ rather than on a single point. This peculiarity is interesting because it arises in modeling various physical questions such as the aeroelastic behavior of suspended flexible cables subjected to icing conditions and wind action (see e.g. [14–16]) or the dust production and diffusion in the fusion devices (see [2–4,11,12,18,19] and the references therein). Moreover, nonlocal nonlinear problems also arise in thermodynamical equilibrium questions (see e.g. [5,7,8,13,21]) and we also cite [9,1].

The results in this paper concern both the analytical and numerical study of the nonlocal problem (1.1). In addition to the analytical study, which aims to find nice sufficient conditions to the existence and uniqueness of the solution, numerical methods for nonlocal nonlinear problems are usually constructed by applying central difference formulas to the derivatives

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of the unknown function and composite quadrature rules (on the same mesh of nodes) to the integrals involving the unknown function.

The main difficulties in this numerical approach consist in solving the nonlocal nonlinear algebraic system deduced from the chosen discretization process, and in proving the convergence of the resulting numerical solution to the analytical one, as the mesh size of the discretized problem tends to zero.

As regards the first difficulty, the solution of the nonlocal nonlinear algebraic system deduced from standard discretization processes, is generally obtained by fixed point iterations assuming that the data fit into the hypotheses of Banach fixed point theorem. The novelty introduced in this paper, consists in using a different fixed point theorem, recently stated by the authors (cf. Theorem 4.1), which allows us to suggest different iterative procedure to get the numerical solution, including fixed point iterations that we prove are globally convergent under less restrictive assumptions than the ones of Banach theorem implying the iteration function is a contractive map. Moreover, as regards the convergence of resulting numerical solution to the analytical one, due to the nonlocal nonlinearity, it cannot be stated by standard technique and in Theorem 3.5 *ghost* sequences have been introduced for the purpose.

We recall that problem (1.1) was studied for the first time by J.R. Cannon and D.J. Galiffa, who, in [8], first proved the existence and uniqueness of the continuous solution, next they proposed a numerical method for its resolution and then they proved the first order convergence. Nevertheless in the last sentence of their paper (see [8, p. 1713]) they observed how their hypothesis $\alpha'(q) < 0$ is sufficient but not necessary for both uniqueness and convergence, leaving open the analysis of the problem in the case it is not satisfied.

In this paper, following the feeling of the authors of [8], "*it is also important to mention that the numerical methods developed in this paper function as paradigm for further numerical methods to be developed* ...", we start from their results with the main aim of answering to the previous open question.

To be more specific, here we prove the unique solvability of (1.1) as well as the convergence of the Cannon and Galiffa method under different hypotheses, which can explain the unexpected good behavior of the numerical solution noted in [8, Section 6] for the case $\alpha(q) = q$, giving an answer to the mentioned open question. Also, under the same hypotheses considered in [8], we state the order of convergence to be equal to two. Moreover, starting from the discretization used in [8], we study alternative algorithms for solving the resulting nonlocal nonlinear discrete problem. Instead of the interval–halving scheme proposed in [8], we suggest to handle the nonlocal nonlinearity of the discrete problem by three different iteration procedures (cf. $(i_1), (i_2), (i_3)$) converging to the numerical solution under suitable assumptions. These conditions cover several cases, so that we can choose from time to time the iteration to apply in dependence of the specific problem we want to solve (see the examples in the last section).

The paper is organized as follows. In Section 2 jointly with some preliminary results, we provide a theorem on the unique solvability of (1.1) and its discrete approximation. Section 3 concerns the convergence estimates and Section 4 proposes some possible iterative algorithms for finding the numerical solution. Finally, Section 5 contains some numerical tests.

2. Preliminaries

Following Cannon and Galiffa [8], we consider the next auxiliary problem

$$-\alpha(q)u''(x) + [u(x)]^{2n+1} = 0, \quad x \in (0,1), \ u(0) = a, \ u(1) = b, \tag{2.2}$$

where $q \in \mathbb{R}$ is a fixed parameter replacing the integral of the unknown function in (1.1).

Moreover, we denote by $u_i := u(x_i)$ the values of the unknown solution u at the grid

$$x_i := ih, \quad i = 0, \dots, N, \ h = \frac{1}{N} > 0 \tag{2.3}$$

and consider the well–known approximation formulas, where $\xi_i, \ \xi \in (0,1)$

$$u''(x_i) = \Delta_h^2 u_i - \frac{h^2}{12}u^{(4)}(\xi_i), \qquad \Delta_h^2 u_i := \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad i = 1, \dots, N-1, \tag{2.4}$$

$$\int_0^1 u(x)dx = Q_h(u) - \frac{h^2}{12}u''(\xi), \quad Q_h(u) := h\left[\frac{u_0 + u_N}{2} + \sum_{i=1}^{N-1} u_i\right]. \tag{2.5}$$

By means of (2.4) and (2.5), we get the next discrete counterparts of (2.2) and (1.1), respectively

$$-\alpha(q)\Delta_h^2 w_i + w_i^{2n+1} = 0, \quad i = 1, \dots, N-1, \ w_0 = a, \ w_N = b, \tag{2.6}$$

$$-\alpha(Q_h(w))\Delta_h^2 w_i + w_i^{2n+1} = 0, \quad i = 1, \dots, N-1, \ w_0 = a, \ w_N = b, \tag{2.7}$$

which can be equivalently written as the following nonlinear systems

$$[\alpha(q)A + D_w]w = \alpha(q)p, \tag{2.8}$$