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Can a semi-simple eigenvalue admit fractional sensitivities?

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A B S T R A C T

We perform high-order sensitivity analysis of eigenvalues and eigenvectors of linear systems depending on parameters. Attention is focused on double not-semi-simple and semi-simple eigenvalues, undergoing perturbations, either of regular or singular type. The use of integer (Taylor) or fractional (Puiseux) series expansions is discussed, and the analysis carried out both on the characteristic polynomial and on the eigenvalue problem. It is shown that semi-simple eigenvalues can admit fractional sensitivities when the perturbations are singular, conversely to the not-semi-simple case. However, such occurrence only manifests itself when a second-order perturbation analysis is carried out. As a main result, it is found that such over-degenerate case spontaneously emerges in bifurcation analysis, when one looks for the boundaries of the stability domain of circulatory mechanical systems possessing symmetries. A four degree-of-freedom system under a follower force is studied as an illustrative example.

1. Introduction

The dynamics of a linear system depend on its eigenvalues $\lambda^{(i)}$, $i = 1, 2, \ldots$. Very often, one is not interested in analyzing the behavior of a specific system, but, rather, that of a family of systems, parametrized by one or more control parameters $\boldsymbol{\mu} \in \mathbb{R}^M$, so that $\lambda = \lambda(\boldsymbol{\mu})$ (apex i omitted from now on). Once the eigenvalues $\lambda_0 := \lambda(\boldsymbol{\mu}_0)$ have been evaluated at a selected point $\boldsymbol{\mu}_0$ of the parameter space, one would predict how the eigenvalues vary in a small ball $\mathcal{R}$ around this point, thus avoiding to repeat the analysis for several values of the increment $\delta\boldsymbol{\mu} := \boldsymbol{\mu} - \boldsymbol{\mu}_0$. The problem is of particular importance in bifurcation analysis, where the eigenvalues decide on stability or instability of the equilibrium points.

The task is usually performed by perturbation methods [1]. These require, in the order: (a) to select a family of 'exploring curves' $\mathcal{C} \in \mathcal{R}$, of parametric equations $\boldsymbol{\mu} = \boldsymbol{\mu}(\varepsilon)$, emanating from $\boldsymbol{\mu}_0 = \boldsymbol{\mu}(0)$, where $0 < \varepsilon \ll 1$ is a perturbation parameter; (b) to assume a (formal) series expansion for the eigenvalue $\lambda(\boldsymbol{\mu}(\varepsilon))$, namely:

$$\lambda = \lambda_0 + \sum_{k=1}^{\infty} \varepsilon^k \lambda_k, \tag{1}$$

(c) to solve in chain several perturbation equations in the unknown coefficients λ_k. These latter are called the 'sensitivities' of the eigenvalue (by understanding 'at $\boldsymbol{\mu}_0$ along $\mathcal{C}$'). Since the series expansion only involves *integer powers* of ε (i.e. it is a

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Taylor series), we will call them *integer sensitivities*. Usually, eigenvalue sensitivities are evaluated together with *eigenvector sensitivities*, since the whole eigenvalue problem is tackled; on the other hand, example exist in literature, in which the perturbation analysis is directly carried out on the characteristic polynomial [1].

However, it is also well known [1–6] that there exist degenerate cases in which λ is *not analytical* at $\boldsymbol{\mu}_0$, so that it does not admit a Taylor expansion. This case occurs when $m = 2, 3, \ldots$ eigenvalues coalesce at $\boldsymbol{\mu}_0$ and, moreover, the geometric multiplicity n of the associated eigenvectors is less than the algebraic multiplicity m. The eigenvalue is then said to be *defective*, in the sense that $m - n$ eigenvectors are missing, so that the eigenspace must be supplemented by generalized eigenvectors. In such degenerate cases, a generalized (or Puiseux) series must be used, involving *fractional powers* of the perturbation parameter. In the most frequent case of not-derogatory matrix (i.e. $n = 1$, which entails the existence of a unique Jordan block associated with λ), this series reads:

$$\lambda = \lambda_0 + \sum_{k=1}^{\infty} \varepsilon^{k/n} \lambda_{k/n}. \tag{2}$$

We will refer to the coefficients $\lambda_{k/n}$ of the series as the *fractional sensitivities* of the eigenvalue (at $\boldsymbol{\mu}_0$ along $\mathcal{C}$). More general cases have been studied in [7–9], by referring to the eigenvalue problem, and in [1] to the characteristic polynomial.

In spite of a naif criticism, according to which such degenerate cases are unimportant, since it is unlikely to meet with them, their study is mandatory in Bifurcation Theory. There, the nature of the singularity determines the dynamics around it, not only in the linear, but even in the nonlinear field! As a matter of fact, a double-zero bifurcation analysis (also known as Takens-Bogdanov bifurcation) calls for unfolding (i.e. for analyzing the effects of a generic perturbation) such a degenerate eigenvalue occurrence (see, e.g., [10–13]). In addition, in the context of non-generic perturbations in relation to stability problems, is relevant [14] and the references there cited.

When coalescence of the eigenvalues occur and the system is generic, the eigenvalue is defective (also said *not-semi-simple*), this entailing the existence of fractional sensitivities. If, in contrast, suitable geometrical symmetries exist or energy conservation holds, the eigenvalue is not-defective (also said *semi-simple*), so that integer sensitivities are admitted, in spite of the coalescence. However, exceptions are known to exist to this rule, concerning defective eigenvalue. Indeed, it is known that there exist *singular directions* outgoing from $\boldsymbol{\mu}_0$, for which the fractional series expansion (2) degenerates in the integer series expansion (1). This over-degenerate occurrence (i.e. not only $\boldsymbol{\mu}_0$ is singular, but even the perturbation is singular) has been highlighted in [15] in analyzing the Ziegler Paradox, and in [11] in studying the double-zero bifurcation. In these paper, it is also stressed that the singular direction posses remarkable properties, namely: (a) it is the (unique) direction in which damping has a beneficial effect on the Beck's load, and (b) it is the tangent to the divergence boundary emanating from $\boldsymbol{\mu}_0$ in the Takens-Bogdanov bifurcation. Hence, not only degeneracies are important, but even over-degeneracies (concerning their unfolding) are important.

It is now quite natural to ask to ourselves if a converse case, which seems not to have been studied in literature, can occur. Namely: (a) *can a semi-simple eigenvalue admit fractional sensitivities*, as an effect of singular perturbations? And, in the affirmative case, (b) *do singularities determine special system behaviors*, making them worthy of study?

This paper gives affirmative answers to both questions. In Section 2 a perturbation analysis is carried out on the characteristic polynomial, to display the occurrence of over-degeneracies, and to investigate the mechanism leading to singularity of the expansion. In Section 3 a second-order analysis for the eigenvalue problem is performed for semi-simple eigenvalues under regular perturbations. In Section 4 a new perturbation algorithm is developed to analyze singular perturbations of semi-simple eigenvalues. In Section 5, the theory is specialized to mechanical circulatory systems (i.e. Hamiltonian systems under non-conservative positional forces), to show the existence of a strict link between singular perturbations and stability boundaries. In Section 6 a numerical example is worked out. Finally, in Section 7, some conclusions are drawn.

2. Characteristic polynomial analysis

Let us consider the linear algebraic eigenvalue problem:

$$(\mathbf{A}(\boldsymbol{\mu}) - \lambda \mathbf{I})\mathbf{w} = \mathbf{0}, \tag{3}$$

in which $\mathbf{A}(\boldsymbol{\mu})$ is a real square matrix depending on a set of real parameters $\boldsymbol{\mu}$, $\mathbf{I}$ the identity matrix, λ is a generally complex eigenvalue and $\mathbf{w}$ the associated eigenvector, respectively. Eq. (3) calls for solving a characteristic equation of the type:

$$F(\lambda, \boldsymbol{\mu}) := \det[\mathbf{A}(\boldsymbol{\mu}) - \lambda \mathbf{I}] = 0. \tag{4}$$

We look for an asymptotic expansion of the unknown $\lambda(\boldsymbol{\mu})$ around the point $\boldsymbol{\mu}_0$ of the parameter space, for which we assume that $\lambda_0 := \lambda(\boldsymbol{\mu}_0)$ is known. To this end, we explore the neighborhood $\mathcal{R}$ of $\boldsymbol{\mu}_0$ by selected curves $\boldsymbol{\mu} = \boldsymbol{\mu}(\varepsilon)$, where $0 < \varepsilon \ll 1$ is a perturbation parameter and $\boldsymbol{\mu}(0) = \boldsymbol{\mu}_0$. Usually, straight lines $\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \varepsilon \boldsymbol{\mu}_1$ are well-suited to the scope, but in some circumstance, to be investigated later, different curves must be used. Along one of these curves, it is $F(\lambda, \boldsymbol{\mu}(\varepsilon))$, so that the characteristic equation reads $F(\lambda, \varepsilon) = 0$, which admits the pair $(\lambda_0, 0)$ as a solution, i.e. $F(\lambda_0, 0) = 0$.