Applied Mathematics and Computation 255 (2015) 207–213

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Extension of van der Corput algorithm to LS-sequences

Ingrid Carbone

University of Calabria, Department of Mathematics, Ponte P. Bucci Cubo 30B, 87036 Arcavacata di Rende, Cosenza, Italy

ARTICLE INFO

Keywords:
Uniform distribution
Sequences of partitions
van der Corput sequences
Low discrepancy

ABSTRACT

The *LS-sequences of points* recently introduced by the author are a generalization of van der Corput sequences. They are constructed by reordering the points of the corresponding *LS-sequences of partitions*. Here we present another algorithm which is simpler to compute than the original construction and coincides with the classical one for van der Corput sequences. This algorithm is based on the representation of natural numbers in base $L+S$. Moreover, when $S \leqslant L$ these sequences have low discrepancy and can be useful in Quasi Monte-Carlo methods.

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1. Introduction

Recently we introduced in [4] the countable class of *LS-sequences of partitions* $\{\rho^n_{L,S}\}$, which belongs to the family of ρ-*refinements* introduced by Aljoša Volčič generalizing Kakutani's splitting procedure: first we divide the unit interval into L "long" intervals having length γ followed by S "short" intervals having length γ^2, where γ is the positive solution of the quadratic equation $Lx + Sx^2 = 1$ with natural coefficients L and S; then we divide all the long intervals proportionally to γ and γ^2, and so on. This way we get a sequence of partitions in which the nth partition is a refinement of the $(n-1)$th one.

In the same paper we presented a recursive formula which reorders the points of any *LS-sequence of partitions* and associates to it a sequence of points, called *LS-sequence of points*. These sequences reduce to the classical *van der Corput sequences* when $S = 0$.

The main interest of the *LS-sequences of points* lies in the fact that, whenever $L \geqslant S$, they have low discrepancy and, for this reason, they are suitable for the implementation of the Quasi Monte Carlo method for computing integrals. If a sequence of points in higher dimension has low discrepancy, it is well known that all the coordinate sequences have low discrepancy. This fact shows the importance of finding new low discrepancy sequences in dimension 1. Similar arguments have been treated in [2,14,24].

The main goal of this paper is to present a new algorithm (explicit and more efficient than the recursive formula mentioned above) to compute the points of *LS-sequences*. Using the representation of natural numbers in base $L+S$, we define a suitable function $\phi_{L,S}$ (called the *LS-radical inverse function*) whose domain is an appropriate countable set $\mathbb{N}_{L,S} \subseteq \mathbb{N}$. This function is a generalization of the *radical inverse function* ϕ_b used to define the classical van der Corput sequences (they coincide when $S = 0$) and is related to the so-called β-*adic Monna map* used to define the β-adic van der Corput sequences. The relation between the latter class of sequences and the *LS-sequences* is interesting, but it will be studied in a forthcoming paper.

E-mail address: i.carbone@unical.it

http://dx.doi.org/10.1016/j.amc.2014.08.063
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2. Preliminaries

Let us begin with some definitions (for a complete overview on uniform distribution and discrepancy see [21,11]).

Definition 2.1. Given a sequence $\{Q_n\}$ of finite subsets of $[0,1[$, with $Q_n = \left\{y_1^{(n)},\dots,y_{t_n}^{(n)}\right\}$, we say that it is *uniformly distributed* if

$$\lim_{n\to\infty}\frac{1}{t_n}\sum_{i=1}^{t_n} f(y_i^{(n)}) = \int_0^1 f(x)\,dx \tag{1}$$

for every continuous (or Riemann integrable) function f defined on $[0,1]$. The *discrepancy* of Q_n is defined by the sequence $\{D(Q_n)\}$, where

$$D(Q_n) = \sup_{0\leqslant a<b\leqslant 1}\left|\frac{1}{t_n}\sum_{j=1}^{t_n}\chi_{[a,b[}(y_j^{(n)}) - (b-a)\right|.$$

We also say that $\{Q_n\}$ has *low discrepancy* if there exists a constant C such that $t_n\, D(Q_n) \leqslant C$.

It is well-known that this bound is optimal and is attained for instance by the Knapowski sequence $\{Q_n\}$ of sets $Q_n = \left\{0,\frac{1}{n+1},\frac{2}{n+1},\dots,\frac{n}{n+1}\right\}$.

Definition 2.2. Given a sequence of points $\{x_n\}$ of $[0,1[$, we say that it is *uniformly distributed* if the sequence $\{Q_n\}$ of sets $Q_n = \{x_1,\dots,x_n\}$ is uniformly distributed. We also say that $\{x_n\}$ has *low discrepancy* if there exists a constant C such that $nD(Q_n) \leqslant C\log n$.

This bound is optimal, too, and is achieved, for instance, by van der Corput sequences.

These definitions can be easily extended to higher dimensions, but the results of this paper concern only the one-dimensional case.

If we consider a sequence $\{\pi_n\}$ of finite partitions of $[0,1[$, with $\pi_n = \left\{[y_i^{(n)},y_{i+1}^{(n)}[,\, 1\leqslant i\leqslant t_n\right\}$, where $y_1^{(n)}=0$ and $y_{t_n+1}^{(n)}=1$, according to the definitions above we say that $\{\pi_n\}$ is *uniformly distributed* if so is the sequence $\{Q_n\}$ of sets $Q_n = \left\{y_1^{(n)},\dots,y_{t_n}^{(n)}\right\}$, and that its discrepancy $D(\pi_n)$ is the discrepancy of $\{Q_n\}$. If there exists a constant C such that $t_nD(\pi_n)\leqslant C$, we say that $\{\pi_n\}$ has low discrepancy. This bound is optimal and is attained for example by the Knapowski sequence of partitions mentioned previously.

Discrepancy gives us the speed with which the averages $\frac{1}{t_n}\sum_{i=1}^{t_n}\chi_{[a,b[}(y_i^{(n)})$ converge to the integral of $f=\chi_{[a,b[}$, the characteristic function of the interval $[a,b[$.

Sequences of partitions and sets have been considered rather early in the theory of uniform distribution. Partition in base 2 has been considered in 1935 by van der Corput [25]. Knapowski [20] studied in 1957 the sequence of partitions of $[0,1[$ into n equal parts and its generalizations.

In the multidimensional case the *LS-sequences of points* have been studied in [5,1].

The sequence of partitions in base b, with $b\in\mathbb{N},\ b\geqslant 2$, is defined by

$$\left\{\left[\frac{i-1}{b^n},\frac{i}{b^n}\right[,\ 1\leqslant i\leqslant b^n\right\}, \tag{2}$$

while the corresponding sequence of points, called the *van der Corput sequence in base b*, is obtained reordering in an appropriate way the points determining them. We will not discuss this aspect in detail now, as this procedure turns out to be a special case of the algorithm we will propose in the last section of this paper.

We only mention Faure's *scrambled van der Corput sequences* [12,13] and the previous extensions to higher dimension by Hammersley [16] and Halton [15]. Hammersley considered sets of N points in $\mathbb{R}^d$ of the kind $\left(x_1^{(n)},x_2^{(n)},\dots x_d^{(n)}\right)$ for $n\leqslant N$, where $x_1^{(n)}=\frac{n}{N}$, while for $2\leqslant k\leqslant d$, $x_k^{(n)}$ is the nth term of the van der Corput sequence of points in base b_{k-1}, the $(k-1)$th prime number. Halton [15] used in higher dimension the b_k-adic van der Corput sequences, where b_k is the kth prime number. The employment of *LS-sequences of points* and also of the related β-adic van der Corput sequences in higher dimension has been initiated in [5,1,17].

From the definitions of uniform distribution of sequences of sets, points and partitions it is clear that, when we want to evaluate the integral of f, the sequences of points provide a more flexible tool, as we can choose in advance any number N of points x_n and afterwards, if we want to improve the approximation, we can add any number of additional points using all the previously calculated values of f.

On the other hand, if we want to increase the number of points and we are dealing with a sequence of sets (or partitions), there are two possibilities. The worst case is when Q_n and Q_{n+k} have few points in common. In this case all the previously calculated values of f have to be thrown away. The situation is better if $Q_n\subset Q_{n+1}$. But even this case has a drawback, as we are limited in the choice of the number of points in which to evaluate the function f, as the averages (1) have no meaning for values between t_n and t_{n+1}.

From this observation stems the main motivation of this paper: given a uniformly distributed sequences of partitions, it is highly desirable to associate to it a uniformly distributed sequence of points having the lowest possible discrepancy. In [4]

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