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Risk management in portfolio applications of non-convex stochastic programming ☆

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ABSTRACT

In this paper, we investigate a method to hedge nonconvex stochastic programming with CVaR constraints and apply the sample average approximation (SAA) method based on bundle method to solve it. Under some moderate conditions, the SAA solution converges to its true counterpart with probability approaching one. This technique is suitable for using by investment companies, brokerage firms, mutual funds, and any business that evaluates risks. It can be combined with analytical or scenario-based methods to optimize portfolios in which case the calculations often come down to non-convex programming. Finally, we illustrate our method by considering several portfolios in the Chinese stocks market.

1. Introduction

The selection of assets or equities is not just a problem of finding attractive investments. Designing the correct portfolio of assets cannot be done by human intuition alone and requires modern, powerful and reliable mathematical programs called optimizers. It is a generally accepted principle portfolio of that a portfolio is designed according to the investor's risk tolerance, time frame and investment objectives. The monetary value of each asset may influence the risk/reward ratio of the portfolio and is referred to as the asset allocation of the portfolio. When determining a proper asset allocation one aims at maximizing the expected return and minimizing the risk.

In Chinese, most people are *Risk Averse* (A description of an investor who, when faced with two investments with a similar expected return (but different risks), will prefer the one with the lower risk). Therefore how to control the risk of portfolio seems to be very important. There are many ways to measure risk, among these VaR ([1,7,18,21], etc.) and CVaR ([2,8,10,13,16,19], etc.) have been proved to be effective.

VaR is a very popular measure of risk, but it has undesirable mathematical characteristics such as a lack of subadditivity and convexity, see Artzner [4,5]. As an alternative measure of risk, CVaR is known to have better properties than VaR, see [4,11]. Pflug [17] proves that CVaR is a coherent risk measure having the following properties: transition equivariant, positively homogeneous and convex.

Stochastic programming models that consider the variability of the uncertain parameters typically optimize the expected performance measures of the problem so as to obtain optimal solution that perform well on average of the uncertain events.

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In risk management, stochastic convex optimization (with CVaR objective or CVaR constraints) is a powerful tool for portfolio, for instance, the *revenue maximization model*

$$\begin{aligned} \min\ & -r(\omega)^T x \\ \text{s.t. } & e^T x = 1, \\ & g(x) - q \leqslant 0, \\ & \mathrm{CvaR}_\alpha[G(x,r)] - p \leqslant 0, \\ & x_i \geqslant 0, i = 1,\ldots,n, \\ & x \in \mathcal{R}^n, \end{aligned} \tag{1.1}$$

where n denotes type of shares we considered, r is a random variable which represent stock returns(per month), $r \in R^n$, the function $g(x)$ represents portfolio risk of n-assets, $x = (x_1,\ldots,x_n)$ represents the amount of money invested in the different assets at the beginning of the examined period, e is a unit vector, $e \in R^n$; $G(x,r) = -r^T x$, $r^T x$ represents the investment income, $-r^T x$ represents the investment losses.

However, in fact when come to CVaR constraints problem whose objective function is often non-convex like *minimum risk model*

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. } & e^T x = 1, \\ & g(x) - q \leqslant 0, \\ & \mathrm{CvaR}_\alpha[G(x,r)] - p \leqslant 0, \\ & x_i \geqslant 0, \quad i = 1,\ldots,n, \\ & x \in \mathcal{R}^n, \end{aligned} \tag{1.2}$$

where $f(x) = x^T \sigma x$, where $\sigma = [\sigma_{i,j}]_{n\times n}$, $\sigma_{i,j} = E[(r_i - Er_i)(r_j - Er_j)]$, $i,j = 1,\ldots,n$. We know that σ may not be positive-definite so (1.2) could be a non-convex programming.

In the past few years, Krokhmal et al. [15] and Rockafellar and Uryasev [19,20] introduce the idea of using CVaR in dynamic models. Krokhmal et al. [15] is also the first paper dealing with CVaR constraints. Fábián [12] proposes decomposition frameworks for handling CVaR objectives and constraint in two-stage stochastic model. Ágoston [3] uses SRA algorithm for solving CVaR minimization problems. Bardou [6] solves CVaR hedging using quantization based stochastic approximation algorithm.

These methods are very useful in stochastic convex optimizations with CVaR objectives and constraints but may not be extend to solve the non-convex problems.In this paper we consider the stochastic model (CVaR constraints) which contains (1.2) as special case

$$\min_{x\in X} \{E[f(x,\xi)] : \mathrm{CvaR}_\alpha[G(x,\omega)] \leqslant q\}, \tag{1.3}$$

where we use random variable ξ and w to reflect randomness of investment market, $f(x,\xi):\ R^n \times \Omega \to R$ is a random function and for every ξ, f may be non-convex. Our works focus on using SAA method which has been widely used in many papers such as [9] and [22] based on bundle method to solve stochastic model (CVaR constraints) (1.3). Bundle methods are recognized as one of the most efficient optimization methods for solving nonsmooth problems. Under some moderate conditions, we show that the deviation of the solution set of SAA problem from the solution set of true problem converges to zero. We also give the numerical results for illustrating the effectiveness of our method.

The rest of the paper is organized as follows. In the next section, we introduce some basic definitions and results to be used in this paper. Convergence analyses of the SAA method are given for constraint set, general non-convex stochastic optimization and non-convex stochastic program with CVaR constraints in Section 3. Numerical experiments and applications in risk management are showed at Section 4. We present our conclusion in Section 5.

2. Preliminaries

In this section, we recall some background materials and preliminary results to be used in this paper.

Let $h(x,\xi)$ be the loss associated with the decision vector x to be chosen from a certain subset X of R^n and the random vector $\xi \in R^m$. For each x, the loss $h(x,\xi)$ is a random variable having a distribution in R induced by that of ξ and $p(\xi)$ denote the density function of ξ. The probability of $h(x,\xi)$ not exceeding a threshold γ is given by

$$\Psi(x,\gamma) = \int_{h(x,\xi)} p(\xi)d\xi. \tag{2.1}$$

As a function of γ for fixed x, $\Psi(x,\xi)$ is the cumulative distribution function for the loss associated with x. The α-VaR and α-CVaR values for the loss random variable associated with x and any specified probability level α in $(0,1)$ will be denoted by