Applied Mathematics and Computation 258 (2015) 12–21

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Asymptotical stability of the exact solutions and the numerical solutions for a class of impulsive differential equations

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ARTICLE INFO

Keywords:
Impulsive delay differential equations
Runge–Kutta method
Stable
Asymptotically stable

ABSTRACT

This paper is concerned with stability and asymptotical stability of a class of impulsive delay differential equations (IDDEs). Stability and asymptotical stability of the system of IDDEs are studied by the properties of delay differential equations (DDEs) without impulsive perturbations. Base on this idea, numerical methods of IDDEs are constructed. Stability and asymptotical stability of numerical methods of IDDEs are also studied by the properties of numerical methods of DDEs.

1. Introduction

In this paper, we consider the impulsive delay differential equation (IDDE):

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau)), & t \geqslant 0, \quad t \neq k\tau, \quad k = 0,1,2,\ldots, \\ \Delta x(t) = r x(t^-), & t = k\tau, \quad k = 0,1,2,\ldots, \\ x(t) = \Phi(t), & t \in [-\tau, 0), \end{cases} \tag{1.1}$$

and the same equation with another initial function:

$$\begin{cases} \tilde{x}'(t) = f(t, \tilde{x}(t), \tilde{x}(t-\tau)), & t \geqslant 0, \quad t \neq k\tau, \quad k = 0,1,2,\ldots, \\ \Delta \tilde{x}(t) = r \tilde{x}(t^-), & t = k\tau, \quad k = 0,1,2,\ldots, \\ \tilde{x}(t) = \tilde{\Phi}(t), & t \in [-\tau, 0), \end{cases} \tag{1.2}$$

where $r \neq -1, \tau > 0, \Phi$ and $\tilde{\Phi}$ are continuous functions on $[-\tau, 0), \lim_{t\to 0^-} \Phi(t)$ and $\lim_{t\to 0^-} \tilde{\Phi}(t)$ exist, $x'(t)$ denotes the right-hand derivative of $x(t)$ and $\Delta x(t) = x(t) - x(t^-)$. Let $\langle \cdot, \cdot \rangle$ be a given inner product on $\mathbb{C}^d$ and $\|\cdot\|$ the corresponding norm. Assume that the function $f : [t_0, \infty) \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ is continuous in t and uniformly Lipschitz continuous with respect to the second and third variable in the following sense: there exist two continuous function $\sigma(t)$ and $\gamma(t)$ such that

$$\sigma(t) \geqslant \sup_{z, y_1, y_2 : y_1 \neq y_2} \frac{\Re(\langle f(t, y_1, z) - f(t, y_2, z), y_1 - y_2 \rangle)}{\|y_1 - y_2\|^2}, \tag{1.3}$$

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http://dx.doi.org/10.1016/j.amc.2015.01.115
0096-3003/

$$\gamma(t) \geqslant \sup_{y,z_1,z_2;z_1\neq z_2} \frac{\|f(t,y,z_1) - f(t,y,z_2)\|}{\|z_1 - z_2\|}. \tag{1.4}$$

Definition 1.1. The system (1.1) is said to be stable, if for any other initial function $\tilde{\Phi}(t)$, there is a positive constant C such that the difference $x(t) - \tilde{x}(t)$ fulfills

$$\|x(t) - \tilde{x}(t)\| \leqslant C \cdot \sup_{t\in[-\tau,0)} \|\Phi(t) - \tilde{\Phi}(t)\|, \quad t \geqslant 0.$$

Definition 1.2. The system (1.1) is said to be asymptotically stable, if for any other initial function $\tilde{\Phi}(t)$, the difference $x(t) - \tilde{x}(t)$ fulfills

$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0.$$

In recent years, stability and asymptotical stability of IDDEs have been widely studied by Lyapunov method or Razumikhin method, for example [2,4–8,11] etc. In this paper, stability and asymptotical stability of (1.1) are studied by the properties of DDEs without impulsive perturbations.

There are lots of papers about the exact solutions of IDDEs. But there are a few papers about the numerical solutions of linear IDDEs. In paper [3], Ding etc studied the convergence property of an Euler method for IDDEs. In paper [12], exponential stability of explicit Euler method for a class of linear IDDEs was studied. In paper [13], new methods for a class linear IDDEs were constructed. The convergence and asymptotic stability of the methods for the linear IDDEs were studied. But to the best of our knowledge, up to now, there are few articles referring to stability of numerical methods for nonlinear impulsive differential delay equations. In this paper, some new different conditions for asymptotical stability of system (1.1) are obtained, moreover, asymptotical stability of numerical methods for system (1.1) is studied under these conditions.

The results of this paper can be considered as an extension to the results of the paper [13] about asymptotic stability of linear IDDEs and also can be considered as an extension to the results of Tian and Kuang [9] and Zennaro [14] about asymptotic stability of DDEs as the delay is a constant.

The rest of this paper is organized as follows. In Section 2, we introduce some results about DDEs. In Section 3, stability and asymptotic stability of the system (1.1) are studied by the properties of DDEs without impulsive perturbations. In Section 4, some sufficient conditions for stability and asymptotic stability of (1.1) are provided respectively. In the last section, we study stability and asymptotic stability of numerical methods for (1.1).

2. Preliminaries

In this section, we introduce some results about DDEs of the form

$$\begin{cases} y'(t) = g(t, y(t), y(t-\tau)), & t \geqslant t_0, \\ y(t) = \psi(t), & t \in [t_0 - \tau,\ t_0], \end{cases} \tag{2.1}$$

where τ is a constant delay and the initial function $\psi(t)$ is continuous on $[t_0 - \tau, t_0]$. Now consider the same equation with another initial function:

$$\begin{cases} \tilde{y}'(t) = g(t, \tilde{y}(t), \tilde{y}(t-\tau)), & t \geqslant t_0, \\ \tilde{y}(t) = \tilde{\psi}(t), & t \in [t_0 - \tau,\ t_0]. \end{cases} \tag{2.2}$$

Definition 2.1 (*See [10]*). The system (2.1) is said to be contractive or dissipative, if for any other initial function $\tilde{\psi}(t)$, the difference $y(t) - \tilde{y}(t)$ fulfills

$$\|y(t) - \tilde{y}(t)\| \leqslant \max_{t\in[-\tau,0]} \|\psi(t) - \tilde{\psi}(t)\|, \quad t \geqslant 0.$$

Definition 2.2 (*See [1,14]*). The system (2.1) is said to be asymptotically stable, if for any other initial function $\tilde{\psi}(t)$, the difference $y(t) - \tilde{y}(t)$ fulfills

$$\lim_{t\to\infty} \|y(t) - \tilde{y}(t)\| = 0.$$

Assume that the function $g : [t_0, \infty) \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ is continuous in t and uniformly Lipschitz continuous with respect to the second and third variable in the following sense: there exist two continuous function $\lambda(t)$ and $\mu(t)$ such that

$$\lambda(t) \geqslant \sup_{z,y_1,y_2;y_1\neq y_2} \frac{\Re(\langle g(t,y_1,z) - g(t,y_2,z), y_1 - y_2\rangle)}{\|y_1 - y_2\|^2}, \tag{2.3}$$