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On polynomial function approximation with minimum mean squared relative error and a problem of Tchebychef

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ABSTRACT

We consider the problem of constructing a polynomial approximation to a function $f(x)$ over the interval $[-1, 1]$ that minimizes the mean squared relative error (MMSRE) over the interval. We establish sufficient conditions for solving the problem. We then consider a classic problem from a paper of Tchebychef and compare his solution to MMSRE, demonstrating that in some cases the latter approach can yield a more appealing solution and one that it is applicable in a number of situations where the Tchebychef approach is not.

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1. Introduction

There is a rich literature on constructing polynomial approximations to a function on a closed interval. A common framework, which we will also adopt, is to consider a function $f(x)$ on the interval $[-1, 1]$ and to construct a polynomial $p(x)$ over that same interval that minimizes some specified error. Most approaches are described by selecting a norm $\|\cdot\|$ and requiring that $p(x)$ be chosen to minimize $\|f(x) - p(x)\|$, in effect minimizing the distance between the function and the polynomial with respect to the induced metric. Two widely used and well-known choices are the L_2 and L_∞ norms which, respectively, yield the least-squares approximation (i.e. the truncated Fourier–Legendre series) and the Tchebychef approximation (often called the minimax approximation).

One limitation of these approaches is that they focus on controlling the magnitude of the error of the approximation (indirectly in the L_2 case) and do not take into account the magnitude of the relative error. This can lead to some unfortunate behavior. Consider, for example, the problem of approximating $f(x) = e^{-4x}$ with a polynomial of degree 3. Fig. 1 shows the result of fitting using both the least-squares and Tchebychef approaches. Although both are close to the target it is disturbing that there are multiple regions where they lie below the x-axis which is generally unacceptable when fitting a function that is strictly positive. We make special note at this point that although the magnitude of the errors is fairly consistent across the interval (and even across the two methods), the relative scale of the errors is not as can be seen in Fig. 2.

This last observation leads us to investigate the construction of polynomial approximations that minimize *relative error* under an appropriate metric. (See Fig. 3).

1.1. Preliminaries

Let $\{\phi_i(x)\}_{i=1}^n$ be a basis for the set of all polynomials of degree $n-1$. And let

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$$p(x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

be the approximating polynomial. At any point x such that $f(x) \neq 0$ the relative error is given by

$$\frac{f(x) - p(x)}{f(x)} = 1 - \frac{p(x)}{f(x)} = 1 - \sum_{i=1}^{n} a_i \frac{\phi_i(x)}{f(x)}$$

It is well known that we can minimize the L_∞ norm of the relative error and a number of algorithms for doing so are available (see [1] for an excellent survey and ample references to these methods). Not surprisingly, this is always possible if the set of functions

$$\left\{ \frac{\phi_i(x)}{f(x)} \right\}_{i=1}^{n} \tag{1}$$

satisfies a Haar condition. Hence a very simple sufficient condition for there to be a unique solution is that $f(x)$ be continuous and nonvanishing on $[-1, 1]$ although it is also possible under other assumptions (a number of interesting generalizations are described in [2]).

We now consider the approximation defined by minimizing the L_2 norm of the relative error, an approach that we shall call the minimum mean-squared relative error (MMSRE) approximation. Provided all of the integrals exist, the mean-squared relative error is given by

$$\frac{1}{2} \int_{-1}^{1} \left\{ 1 - \sum_{i=1}^{n} a_i \frac{\phi_i(x)}{f(x)} \right\}^2 dx \tag{2}$$

Let

$$\mathbf{a} = [\, a_1 \quad a_2 \quad \ldots \quad a_n \,]^T$$

and consider the $n \times n$ matrix H whose elements are given by

$$H_{i,j} = \int_{-1}^{1} \frac{\phi_i(x)\phi_j(x)}{f^2(x)} dx \tag{3}$$

and the vector $\mathbf{b}$ whose elements are given by

$$b_i = \int_{-1}^{1} \frac{\phi_i(x)}{f(x)} dx \tag{4}$$

Eq. (2) becomes

$$\frac{1}{2} \mathbf{a}^T \{ H\mathbf{a} - 2\mathbf{b} \} + 1 \tag{5}$$

It is clear that this quadratic form is minimized by solving the linear system (i.e. the normal equations)

$$H\mathbf{a} = \mathbf{b} \tag{6}$$

We observe that under the power basis H is a Hankel matrix whose elements are given by

$$H_{i,j} = \int_{-1}^{1} \frac{x^{i+j-2}}{f^2(x)} dx$$

and the vector $\mathbf{b}$ is given by

$$b_i = \int_{-1}^{1} \frac{x^{i-1}}{f(x)} dx$$

It is also worth noting at this point that we may substantially ease the restrictions on $f(x)$ since the only requirements are that integrals (3) and (4) exist, and that H be non-singular. We note that this still requires that $f(x)$ not vanish over the interval but loosens the need for continuity since it is now possible for $f(x)$ to have a finite number of simple poles in the interval which we will find quite useful.

2. A problem of Tchebychef

In 1892 Tchebychef wrote a delightful paper [3] where he considered the problem of best degree $n-1$ polynomial approximation for the function

$$f(x) = \frac{1}{\alpha - x}$$