Applied Mathematics and Computation 258 (2015) 60–66

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Oscillations of difference equations with non-monotone retarded arguments

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ARTICLE INFO

Keyword:
Difference equation
Non-monotone argument
Retarded argument
Oscillatory solutions
Nonoscillatory solutions

ABSTRACT

Consider the first-order retarded difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0$$

where $(p(n))_{n\geqslant 0}$ is a sequence of nonnegative real numbers, and $(\tau(n))_{n\geqslant 0}$ is a sequence of integers such that $\tau(n) \leqslant n-1$, $n \geqslant 0$, and $\lim_{n\to\infty}\tau(n) = \infty$. Under the assumption that the retarded argument is non-monotone, a new oscillation criterion, involving lim inf, is established. An example illustrates the case when the result of the paper implies oscillation while previously known results fail.

1. Introduction

Consider the retarded difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0, \tag{E}$$

where $(p(n))_{n\in\mathbb{N}_0}$ is a sequence of nonnegative real numbers, and $(\tau(n))_{n\in\mathbb{N}_0}$ is a sequence of integers such that

$$\tau(n) \leqslant n-1 \quad \text{for} \quad n \geqslant 0 \quad \text{and} \quad \lim_{n\to\infty}\tau(n) = \infty. \tag{1.1}$$

Here, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$.

Define

$$k = -\min_{n\geqslant 0}\tau(n).$$

(Clearly, k is a positive integer.)

By a solution of the difference equation (E), we mean a sequence of real numbers $(x(n))_{n\geqslant -k}$ which satisfies (E) for all $n \geqslant 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geqslant -k}$ of (E) which satisfies the initial conditions $x(-k) = c_{-k},\ x(-k+1) = c_{-k+1}, \ldots,\ x(-1) = c_{-1},\ x(0) = c_0$.

A solution $(x(n))_{n\geqslant -k}$ of the difference equation (E) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

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http://dx.doi.org/10.1016/j.amc.2015.01.110
0096-3003/

The problem of establishing sufficient conditions for the oscillation of all solutions of (E) have been the subject of many investigation. See [1–16] and the references cited therein. Most of these papers are concerning the special case of the retarded difference equation (E) with monotone argument, while a small number are dealing with the general case of the retarded difference equation (E), in which the retarded argument $(\tau(n))_{n\geqslant 0}$ is non-monotone.

In 1998, Zhang and Tian [16], studied the equation (E) and proved that, if

$$\limsup_{n\to\infty} p(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e} \tag{1.2}$$

then all solutions of (E) oscillate.

In 2006, Chatzarakis, Koplatadze and Stavroulakis [2,3], studied the equation (E) and proved that, if one of the following conditions

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > 1, \quad \text{where } h(n) = \max_{0\leqslant s\leqslant n} \tau(s),\ n \geqslant 0, \tag{1.3}$$

or

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) < \infty \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e} \tag{1.4}$$

is satisfied, then all solutions of (E) oscillate.

Assume that the argument $(\tau(n))_{n\geqslant 0}$ is non-monotone. Set

$$h(n) := \max_{s\leqslant n} \tau(s),\ n \geqslant 0. \tag{1.5}$$

Clearly, h is nondecreasing, and $\tau(n) \leqslant h(n) \leqslant n-1$ for all $n \geqslant 0$.

In 2011, Braverman and Karpuz [1], proved that, if

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1, \tag{1.6}$$

then all solutions of (E) oscillate.

The consideration of non-monotone arguments other than the pure mathematical interest, it approximates the natural phenomena described by equation of the type (E). That is because there are always natural disturbances (e.g. noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone argument becomes non-monotone almost always. In view of this, an interesting question arising in case where the argument $(\tau(n))_{n\geqslant 0}$ is non-monotone, is whether we can state an oscillation criterion involving lim inf.

In the present paper a positive answer to the above question is given.

2. Main result

Theorem 2.1. *Assume that (1.1) holds, and*

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > \frac{1}{e}, \tag{2.1}$$

where $h(n)$ is defined by (1.5). Then all solutions of (E) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $(x(n))_{n\geqslant -k}$ of (E). Since $-(x(n))_{n\geqslant -k}$ is also a solution of (E), we can confine our discussion only to the case where the solution $(x(n))_{n\geqslant -k}$ is eventually positive. Then there exists $n_1 > -k$ such that $x(n),\ x(\tau(n)),\ x(h(n)) > 0$, for all $n \geqslant n_1$. Thus, from (E) we have

$$\Delta x(n) = -p(n)x(\tau(n)) \leqslant 0, \quad \text{for all} \quad n \geqslant n_1,$$

which means that x is an eventually nonincreasing sequence of positive numbers.

Set

$$b(n) = \left(\frac{n-h(n)}{n-h(n)+1}\right)^{n-h(n)+1}, \quad n \geqslant 1. \tag{2.2}$$

Clearly