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On sufficient conditions ensuring the norm convergence of an iterative sequence to zeros of accretive operators

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ABSTRACT

Given two real sequences (r_n) and (α_n), we study the iterative scheme: $x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}x_n$, for finding a zero of an accretive operator A, where u is a fixed element and J_{r_n} denotes the resolvent of A. To ensure its convergence, the real sequence (r_n) is always assumed to satisfy $\sum_{n=0}^{\infty}|r_{n+1}-r_n| < \infty$. In this paper we show this condition can be completely removed, which enables us to improve a result recently obtained by Saejung.

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1. Introduction

Let X be a real Banach space, $A : X \to 2^X$ an accretive operator and J_r the resolvent of A. In the present paper we consider the problem for finding a zero of an accretive operator, that is, find $\hat{x}$ so that $0 \in A\hat{x}$. Given a positive sequence (r_n), let us define an iterative scheme

$$x_{n+1} = J_{r_n}x_n, \quad n = 0, 1, 2, \ldots \tag{1}$$

In Hilbert spaces this scheme is known as the proximal point algorithm (PPA) [8]. It is also known that PPA converges weakly and generally does not necessarily converge strongly (cf. [3]). So it is of special interest to modify the PPA so that the strong convergence is guaranteed. One modification of PPA has the following scheme:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}x_n, \tag{2}$$

where $u \in X$ is fixed and $(\alpha_n) \subseteq (0,1)$ and $(r_n) \subseteq (0,\infty)$ are real sequences. This scheme was first introduced in Hilbert spaces and then extended to Banach spaces by several authors (see e.g. [1,2,7,9,11,14]). In uniformly convex Banach spaces, Aoyama et al. [1] established the strong convergence of (2) provided that the following are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty, 0 < \hat{r} \leqslant r_n \leqslant \bar{r}$;
(C2) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ or $\lim_{n\to\infty}\alpha_n/\alpha_{n+1} = 1$;
(C3) $\sum_{n=0}^{\infty}|r_{n+1}-r_n| < \infty$.

Recently Saejung [9] observed that the resolvent is in fact strongly nonexpansive in uniformly convex spaces, from which he showed that condition (C2) is however superfluous. In this paper, we shall show that condition (C3) is also superfluous. More precisely, we shall prove that condition (C1) is sufficient enough to ensure the strong convergence of algorithm (2). This makes the choice on (r_n) more flexible than before.

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2. Preliminary

Throughout this paper, $(X, \|\cdot\|)$ is a real Banach space with dual space X^*. We use $\langle\cdot,\cdot\rangle$ to denote the pair between X^* and X, that is, $\langle x^*, x\rangle$ is the value of x^* at $x, x^*(x)$. For an operator $A, \mathcal{D}(A)$ stands for the domain of A, and $\mathcal{R}(A)$ the range of A. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote, respectively, the strong and weak convergence to x of the sequence (x_n).

The *normalized duality mapping* $J : X \to 2^{X^*}$ is defined as

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|\}.$$

For every $x, y \in X$, there holds the *subdifferential inequality*:

$$\|x+y\| \leqslant \|x\|^2 + \langle y, j(x+y)\rangle, j(x+y) \in J(x+y).$$

Let U denote the unit sphere of X, that is, $U = \{x \in X : \|x\| = 1\}$. The *modulus of convexity* of X is the function $\delta_X(\varepsilon) : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in U,\ \|x-y\| \geqslant \varepsilon\right\}.$$

X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$; *smooth* if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists for all $x, y \in U$. The norm of X is said to be *uniformly Gâteaux differentiable* if the limit in (3) exists for all $x, y \in U$ and its values is attained uniformly for all $x, y \in U$. It is known that X is smooth if and only if J is single-valued; moreover if X has a uniformly Gâteaux differentiable norm then J is norm-to-weak* uniformly continuous on bounded sets of X.

Lemma 1 [13]. *X is uniformly convex if and only if for each* $r > 0$ *there exists a continuous, strictly increasing and convex function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(0) = 0$ *such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leqslant \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\psi(\|x-y\|) \tag{4}$$

for all $x, y \in X$ *with* $\max(\|x\|, \|y\|) \leqslant r$ *and* $0 \leqslant \alpha \leqslant 1$.

Let C be a nonempty closed convex subset of X. An operator $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. From now on, we use $\mathrm{Fix}(T)$ to denote the fixed point set of T, that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Given two subsets D and C of X such that $D \subset C$. An operator $Q : C \to D$ is a *retraction* of C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is *sunny* if, for each $x \in C$ and $t \in [0,1]$, we have

$$Q(tx + (1-t)Qx) = Qx,$$

whenever $tx + (1-t)Qx \in C$, and *sunny nonexpansive* if it is both sunny and nonexpansive. In a smooth Banach space, a retraction $Q : C \to D$ is sunny and nonexpansive, if and only if

$$\langle x - Q(x), J(z - Q(x))\rangle \leq 0, \quad \forall x \in C, z \in D. \tag{5}$$

Lemma 2 ([1,9]). *Let X be a uniformly convex Banach space with a uniformly Gâteaux differential norm and* $T : C \to C$ *a nonexpansive operator with* $\mathrm{Fix}(T) \neq \emptyset$*. If* (x_n) *is a bounded sequence in C such that* $\|(I-T)x_n\| \to 0$*, then*

$$\overline{\lim}_{n\to\infty}\langle u - z, J(x_n - z)\rangle \leqslant 0,$$

where $z \in \mathrm{Fix}(T)$ *satisfies* $\langle u - z, J(y - z)\rangle \leqslant 0,\ \forall y \in \mathrm{Fix}(T)$*, that is,* $z = Q(u)$ *with* $Q : C \to \mathrm{Fix}(T)$ *the unique sunny nonexpansive retraction.*

An operator $A : X \to 2^X$ is said to be *accretive* if, for every $x, y \in \mathcal{D}(A)$, there holds

$$\|x - y\| \leqslant \|x - y + t(u - v)\|, \tag{6}$$

for all $u \in Ax,\ v \in Ay,\ t > 0$. It is known that A is accretive if and only if for each $x, y \in \mathcal{D}(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geqslant 0,$$

for all $u \in Ax,\ v \in Ay$. In what follows, we assume that A is an accretive operator. Then we can define, for each $r > 0$, the resolvent $J_r : \mathcal{R}(I + rA) \to \mathcal{D}(A)$ by $J_r := (I + rA)^{-1}$, and the Yosida approximation $A_r : \mathcal{R}(I + rA) \to \mathcal{R}(A)$ by $A_r = (I - J_r)/r$. Denote by $A^{-1}(0)$ the zero set of A. It is well known that J_r is single-valued and nonexpansive, $\mathrm{Fix}(J_r) = A^{-1}(0)$ and $A_r x \in A(J_r x)$ for every $x \in \mathcal{R}(I + rA)$.

The following two lemmas is due to [4]. We present the proof for the completeness.

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