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Note on the complexity of deciding the rainbow (vertex-) connectedness for bipartite graphs

Shasha Li [a], Xueliang Li [b], Yongtang Shi [b,*]

[a] Department of Fundamental Course Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, PR China
[b] Center for Combinatorics and LPMC-TJKLC Nankai University, Tianjin 300071, PR China

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ABSTRACT

A path in an edge-colored graph is said to be a rainbow path if no two edges on the path share the same color. An edge-colored graph is (strongly) rainbow connected if there exists a rainbow (geodesic) path between every pair of vertices. The (strong) rainbow connection number of G, denoted by ($scr(G)$, respectively) $rc(G)$, is the smallest number of colors that are needed in order to make G (strongly) rainbow connected. A vertex-colored graph G is rainbow vertex-connected if any pair of vertices in G are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection number of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. Though for a general graph G it is NP-Complete to decide whether $rc(G) = 2$ (or $rvc(G) = 2$), in this paper, we show that the problem becomes easy when G is a bipartite graph. Whereas deciding whether $rc(G) = 3$ (or $rvc(G) = 3$) is still NP-Complete, even when G is a bipartite graph. Moreover, it is known that deciding whether a given edge(vertex)-colored (with an unbound number of colors) graph is rainbow (vertex-) connected is NP-Complete. We will prove that it is still NP-Complete even when the edge(vertex)-colored graph is bipartite. We also show that a few NP-hard problems on rainbow connection are indeed NP-Complete.

1. Introduction

We follow the terminology and notations of [2] and all graphs considered here are finite and simple.

As a means of strengthening the connectivity, Chartrand et al. in [5] first introduced the concepts of rainbow connection and strong rainbow connection. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path P in G is a *rainbow path* if no two edges of P are colored the same. The graph G is *rainbow connected* (*with respect to* c) if G contains a rainbow $u - v$ path for any pair of vertices u and v of G. In this case, the coloring c is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring*. The *rainbow connection number* of G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. A *rainbow $u - v$ geodesic* in G is a *rainbow $u - v$ path* of length $d(u, v)$, where $d(u, v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u - v$ geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring* of G. Similarly, we define the *strong rainbow connection number* of

* Corresponding author.
E-mail addresses: lss@nit.zju.edu.cn (S. Li), lxl@nankai.edu.cn (X. Li), shi@nankai.edu.cn (Y. Shi).

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a connected graph G, denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected. Clearly, we have $diam(G) \leqslant rc(G) \leqslant scr(G) \leqslant m$, where $diam(G)$ denotes the diameter of G and m is the number of edges of G. Moreover, it is easy to verify that $src(G) = rc(G) = 1$ if and only if G is a complete graph, that $rc(G) = 2$ if and only if $src(G) = 2$, and that $rc(G) = n - 1$ if and only if G is a tree.

Similar to the concept of rainbow connection number, Krivelevich and Yuster [10] proposed the concept of rainbow vertex-connection number. Let G be a nontrivial connected graph with a vertex-coloring $c : V(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$. A path P in G is *rainbow vertex-connected* if its internal vertices have distinct colors. The graph G is *rainbow vertex-connected* (*with respect to* c) if any pair of vertices are connected by a rainbow vertex-connected path. In this case, the coloring c is called a *rainbow vertex-coloring* of G. If k colors are used, then c is a *rainbow k-vertex-coloring* and G is *rainbow k-vertex-connected*. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. It is easy to observe that if G is of order n then $rvc(G) \leqslant n - 2$, $rvc(G) = 0$ if and only if G is a complete graph, and $rvc(G) = 1$ if and only if $diam(G) = 2$. Notice that $rvc(G) \geqslant diam(G) - 1$ with equality if the diameter is 1 or 2. For more results on rainbow connection and rainbow vertex-connection, we refer to the survey [8] and the book [9].

The computational complexity of rainbow (vertex-) connection number has been studied extensively. In [3], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. In [4], Chakraborty et al. confirmed this conjecture. In [1], the complexity of computing $rc(G)$ and $src(G)$ was studied further. It was shown that given any natural number $k \geqslant 3$ and a graph G, it is NP-hard to determine whether $rc(G) \leqslant k$. Moreover, for $src(G)$, it was shown that given any natural number $k \geqslant 3$ and a graph G, determining whether $src(G) \leqslant k$ is NP-hard even when G is bipartite. In this paper, we will point out that the problems in [1] are, in fact, NP-Complete. Though for a general graph G it is NP-Complete to decide whether $rc(G) = 2$ [4], we show that the problem becomes easy when G is a bipartite graph. Whereas deciding whether $rc(G) = 3$ is still NP-Complete, even when G is a bipartite graph.

For the rainbow vertex-connection number, Chen et al. [7] showed that for a graph G, deciding whether $rvc(G) = 2$ is NP-Complete. Recently, Chen et al. [6] obtained a more general result: for any fixed integer $k \geqslant 2$, to decide whether $rvc(G) \leqslant k$ is NP-Complete. In this paper, we continue focusing on the bipartite graph. Similarly, we obtain that deciding whether $rvc(G) = 2$ can be solved in polynomial time, whereas deciding whether $rvc(G) = 3$ is still NP-Complete when G is a bipartite graph.

Moreover, it is NP-Complete to decide whether a given edge-colored (with an unbound number of colors) graph is rainbow connected [4] and it is also NP-Complete to decide whether a given vertex-colored graph is rainbow vertex-connected [7]. We will prove that the two problems are still NP-Complete even when the graph is bipartite.

2. Main results

At first, we restate several results in [4,1].

Lemma 1 [4]. *Given a graph G, deciding if* $rc(G) = 2$ *is NP-Complete. In particular, computing* $rc(G)$ *is NP-Hard.*

Lemma 2 [1]. *For every* $k \geqslant 3$*, deciding whether* $rc(G) \leqslant k$ *is NP-Hard.*

Lemma 3 [1]. *Deciding whether the rainbow connection number of a graph is at most 3 is NP-Hard even when the graph G is bipartite.*

Lemma 4 [1]. *For every* $k \geqslant 3$*, deciding whether* $src(G) \leqslant k$ *is NP-Hard even when G is bipartite.*

We will show that "NP-hard" in the above results can be replaced by "NP-Complete" if k is any fixed integer. It suffices to show that these problems belong to the class NP for any fixed k. In fact, from the proofs in [1], for the problems in Lemmas 2 and 4, "For every $k \geqslant 3$" can be replaced by "For any fixed $k \geqslant 3$".

Theorem 1. *For any fixed* $k \geqslant 2$*, given a graph G, deciding whether* $rc(G) \leqslant k$ *is NP-Complete.*

Proof. By Lemmas 1 and 2, it will suffice to show that the problem in Lemma 2 belongs to the class NP. Therefore, if given any instance of the problem whose answer is 'yes', namely a graph G with $rc(G) \leqslant k$, we want to show that there is a certificate validating this fact which can be checked in polynomial time.

Obviously, a rainbow k-coloring of G means that $rc(G) \leqslant k$. For checking a rainbow k-coloring, we only need to check whether k colors are used and for any two vertices u and v of G, whether there exists a rainbow $u - v$ path. Notice that for two vertices u, v, there are at most n^{l-1} $u - v$ paths of length l, since if let $P = ut_1t_2 \ldots t_{l-1}v$, there are less than n choices for each t_i $(i \in \{1, 2, \ldots, l-1\})$. Therefore, G contains at most $\Sigma_{l=1}^{k} n^{l-1} \leqslant kn^{k-1} \leqslant n^k$ $u - v$ paths of length no more than k. Then