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A family of bivariate rational Bernstein operators

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A B S T R A C T

Rational Bernstein operators are widely used in approximation theory and geometric modeling but in general they do not reproduce linear polynomials. Based on the work of P. Piţul and P. Sablonnière, we construct a new family of triangular and tensor product bivariate rational Bernstein operators, which are positive and reproduce the linear polynomials. The main result is a proof of convergence of the bivariate rational Bernstein operators defined on the square or triangle.

1. Introduction

Let $B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}$ be the Bernstein basis of degree n for $i = 0, 1, \ldots, n$, and let

$$\mathcal{B}_n f(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_i^n(x) \tag{1}$$

be the Bernstein operator defined for $f \in C([0,1])$, the space of all continuous functions on the unit interval $[0,1]$. Bernstein operators were firstly presented by S.N. Bernstein in [13] for proving the Weierstrass Approximation Theorem. Bernstein operators have been investigated by many mathematicians, see [1–5,9,10,12], and they have several applications in Computer Aided Geometric Design, see [1,6,11,12]. For the history of theory and applications of Bernstein operators, we refer to Farouki's paper [12]. It is standard fact that $\mathcal{B}_n m_r = m_r$ for $r = 0, 1$, where m_r is defined as $m_r(x) = x^r$ for $x \in [0,1]$ and $r \in \mathbb{N}_0$, that is, the operator $\mathcal{B}_n$ reproduces linear polynomials.

The rational Bernstein operator is defined by the expression

$$\mathcal{Q}_n f(x) = \sum_{i=0}^{n} \tilde{\omega}_i f\left(\frac{i}{n}\right) \frac{B_i^n(x)}{Q_n(x)}, \quad x \in [0,1], \tag{2}$$

where $\tilde{\omega}_i, i = 0, 1, \ldots, n$, are are positive numbers and $Q_n(x) \in \mathbb{P}_n$ is a given polynomial of degree less or equal to n which is strictly positive over $[0,1]$. It is easy to achieve that $\mathcal{Q}_n$ reproduces the constant function m_0 by requiring that

$$Q_n(x) = \sum_{i=0}^{n} \tilde{\omega}_i B_i^n(x). \tag{3}$$

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In contrast to the classical case (1), the rational Bernstein operators are less studied, in particular no general convergence result is to be expected to hold. Simple examples show that $\mathcal{Q}_n$ does not reproduce linear polynomials under the general assumptions (2) and (3). However, Piţul and Sablonnière (see [1]) introduced a special class of univariate rational Bernstein operators of the form

$$\mathcal{LR}_n f(x) = \frac{\sum_{i=0}^{n} \bar{\omega}_i f(x_i^{(n)}) B_i^n(x)}{Q_{n-1}(x)}, \quad x \in [0,1] \tag{4}$$

where the positive numbers $\bar{\omega}_i$ and the abscissae $x_i^{(n)}$ for $i = 0, 1, \ldots, n$ are chosen in a way such that $\mathcal{LR}_n$ reproduces linear polynomials. In order to achieve this, it is required that $Q_{n-1}(x) \in \mathbb{P}_{n-1}$, that is

$$Q_{n-1}(x) = \sum_{i=0}^{n-1} \omega_i B_i^{n-1}(x) \tag{5}$$

with given positive numbers $\omega_i, i = 0, 1, \ldots, n-1$. Then the rational Bernstein operator $\mathcal{LR}_n$ defined by (4) reproduces linear polynomials if $\bar{\omega}_i$ and $x_i^{(n)}$ are given by the formulae

$$\bar{\omega}_i = \frac{i}{n}\omega_{i-1} + \left(1 - \frac{i}{n}\right)\omega_i, \quad x_i^{(n)} = \frac{i}{n}\frac{\omega_{i-1}}{\bar{\omega}_i}, \quad 1 \leqslant i \leqslant n-1.$$

Clearly $\bar{\omega}_i$ are positive numbers but it should be noted that the abscissae $x_i^{(n)}$ are only increasing in the variable i under additional assumptions for the coefficients ω_i for $i = 0, 1, \ldots, n$, see e.g. [1,9].

Piţul and Sablonnière (see [1]) proved many interesting facts of this kind of rational Bernstein operator (4), among them a convergence result for $\mathcal{LR}_n$ under the assumption that

$$Q_{n-1}(x) = \mathcal{B}_{n-1}\varphi(x) \tag{6}$$

for a given function $\varphi \in C([0,1])$. Recently, H. Render (see [10]) removed the special requirement (6) and formulated convergence results under the assumption that

$$\Delta_n = \sup_{i=0,\ldots,n-1} \left| x_{i+1}^{(n)} - x_i^{(n)} \right|$$

converges to 0.

The tensor product rational Bernstein operator over $D = [0,1] \times [0,1]$ is generally defined by

$$\mathcal{Q}_{n,m} f(x,y) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} \tilde{\omega}_{i,j} f(\frac{i}{n}, \frac{j}{m}) B_i^n(x) B_j^m(y)}{\sum_{i=0}^{n}\sum_{j=0}^{m} \tilde{\omega}_{i,j} B_i^n(x) B_j^m(y)}, \quad (x,y) \in D. \tag{7}$$

It is easy to see that the operator $\mathcal{Q}_{n,m} f$ reproduces the constants. But in general it does not reproduce the linear polynomials x and y.

The aim of the paper is to extend the results of Piţul and Sablonnière to the rational Bernstein operators defined on the square $D = [0,1] \times [0,1]$ and on the triangle. In the following sections we will restrict the discussion to the case of a square which is technically easier and omit the proofs to the case of triangle and provided only the essential formulae.

To achieve the reproduction of linear polynomials, we assumed that the polynomial

$$Q_{n-1,m-1}(x,y) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \omega_{i,j} B_i^{n-1}(x) B_j^{m-1}(y)$$

has positive coefficients $\omega_{i,j}$ and, in analogy to the construction of Piţul and Sablonnière (see [1]), we set

$$\bar{\omega}_{i,j} = \frac{i}{n}\frac{j}{m}\omega_{i-1,j-1} + \frac{i}{n}\left(1 - \frac{j}{m}\right)\omega_{i-1,j} + \left(1 - \frac{i}{n}\right)\frac{j}{m}\omega_{i,j-1} + \left(1 - \frac{i}{n}\right)\left(1 - \frac{j}{m}\right)\omega_{i,j} \tag{8}$$

for $0 \leqslant i \leqslant n, 0 \leqslant j \leqslant m$, where

$$\omega_{-1,-1} = \omega_{i,-1} = \omega_{-1,j} = 0, \quad 0 \leqslant i \leqslant n, \quad 0 \leqslant j \leqslant m.$$

We construct a family of tensor product rational Bernstein operators $\mathcal{R}_{n,m}$ over $D = [0,1] \times [0,1]$ for $f \in C(D)$ by

$$\mathcal{R}_{n,m} f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} \bar{\omega}_{i,j} f(x_{i,j}, y_{i,j}) \frac{B_i^n(x) B_j^m(y)}{Q_{n-1,m-1}(x,y)} = \frac{P_{n,m}(x,y)}{Q_{n-1,m-1}(x,y)}, \tag{9}$$