Applied Mathematics and Computation 258 (2015) 172–181

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Determinantal representations of hyperbolic forms via weighted shift matrices

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ARTICLE INFO

Keywords:
Determinantal representation
Hyperbolic form
Weighted shift matrix
Numerical range

ABSTRACT

We characterize hyperbolic ternary forms of degrees 3,4, and 5 that admit determinantal representations via cyclic weighted shift matrices.

1. Introduction

Let A be an $n \times n$ matrix. The numerical range of A is defined as the set

$$W(A) = \{\xi^* A \xi : \xi \in \mathbf{C}^n, \xi^* \xi = 1\}.$$

The numerical range $W(A)$ is one of the most well known set-valued characteristics invariants under the unitary equivalence $A \to UAU^{-1}$ for any unitary matrix U. A well known key result due to Toeplitz-Hausdorff shows that $W(A)$ is a compact convex subset of the plane $\mathbb{C}$. Kippenhahn [16] characterized the range $W(A)$ as the convex hull of the real affine part of the dual curve of the algebraic curve $F_A(1, x, y) = 0$ defined by a real ternary form associated with A:

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$.

By using the duality between the compact convex set and its polar set, a description of the range $W(A)$ is reduced to the convex connected component of the open set

$$\{(x, y) \in \mathbb{R}^2 : F_A(1, x, y) \neq 0\}$$

containing $(0, 0)$. Fiedler [9] posed a problem on the characterization of the convex set $W(A)$ by using the duality. He conjectured that if $F(t, x, y)$ is a real hyperbolic ternary form with $F(1, 0, 0) = 1$, then there exist $n \times n$ hermitian matrices H and K satisfying

$$F(t, x, y) = \det(tI_n + xH + yK).$$

Recall that a real ternary form $F(t, x, y)$ of degree n with $F(1, 0, 0) \neq 0$ for which the equation $F(t, \cos\theta, \sin\theta) = 0$ has n real solutions for every $\theta \in [0, 2\pi]$ is said to be *hyperbolic* with respect to $(1, 0, 0)$. The notion of the hyperbolicity was originally

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1 Partially supported by Taiwan National Science Council under NSC 102-2115-M-031-001.
2 Supported in part by Japan Society for Promotion of Science, KAKENHI (23540180).

http://dx.doi.org/10.1016/j.amc.2015.02.019
0096-3003/

introduced for the well posedness of the initial value problem of partial differential equations (cf. [19]). A typical hyperbolic differential operator, corresponding to a solid wave, is given by

$$\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}.$$

Lax [17] conjectured the symbols of hyperbolic differential operators in a similar way. His conjecture is slightly stronger than that of Fiedler, the hermitians matrices H,K are constrained to real symmetric matrices. In the paper [10], Fiedler proved that the Lax conjecture is true for the case that $F(t,x,y)=0$ is an irreducible rational curve. Henrion [13] proved a similar result by using Bezoutian elimination method. Helton and Vinnikov [15] proved that the Lax conjecture is true. Fiedler [10] and Helton-Vinnikov [15] also respectively provided an explicit formula for the real symmetric matrices H,K for the ternary form $F(t,x,y)$. Recently, Plaumann and Vinzant [20] gave an elementary proof of the existence of hermitian matrices H,K for the hyperbolic form $F(t,x,y)$, and hence the validity of the Fidler conjecture. An alternative proof of the Helton-Vinnikov theorem is given in [12]. Another method for the determinantal representations is developed in [18]. The authors of this paper [2,3] provided explicit constructions of determinantal representations of trigonometric polynomial curves.

For arbitrary complex numbers $a_1,a_2,\dots,a_n$, a cyclic weighted shift matrix $S(a_1,a_2,\dots,a_n)$ is an $n\times n$ matrix of the following form

$$S=S(a_1,a_2,\dots,a_n)=\begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & 0\\ 0 & 0 & a_2 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \vdots & \vdots & \ddots & a_{n-1}\\ a_n & 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

The numerical range of a cyclic weighted shift matrix has been widely studied, see, for instance, [1,8,11,14,21,22]. Suppose $a_1,a_2,\dots,a_n$ are non-zero real numbers. It is shown in [4] that the ternary form $F_S(t,x,y)$ associated with a cyclic weighted shift matrix $S(a_1,\dots,a_n)$ satisfies the following conditions:

(i) $F_S(t,x,y)$ is hyperbolic with respect to $(1,0,0)$ and $F_S(1,0,0)=1$,

(ii) $F_S(t,x,y)$ is weakly circular symmetric, that is,

$$F_S(t,\cos(2\pi/n)x-\sin(2\pi/n)y,\sin(2\pi/n)x+\cos(2\pi/n)y)=F_S(t,x,y),$$

(iii) $F_S(t,x,y)$ is symmetric with respect to the line $y=0$, that is, $F_S(t,x,-y)=F_S(t,x,y)$

(iv) $F_S(t,x,y)$ satisfies $F_S(t,-1,-i)=F_S(t,-1,i)=t^n-a$ for some non-zero real number a.

In this paper, we are interested in asking the converse part concerned with the determinantal representation of a ternary form via some cyclic weighted shift matrix. More precisely, for a ternary form $F(t,x,y)$ of degree n satisfying conditions (i)-(iv), does there exist a cyclic weighted shift matrix $S(a_1,\dots,a_n)$ so that $F(t,x,y)=F_S(t,x,y)$? In [6], it shows that the projective algebraic curve

$$C_F=\{[(t,x,y)]\in \mathbf{CP}^2 : F(t,x,y)=0\}$$

defined by a ternary form $F(t,x,y)$ satisfying the above conditions (i)-(iv) has no imaginary singular point and all singular points of C_F are ordinary real nodes. A point (t_0,x_0,y_0) of the curve $F(t,x,y)=0$ is called a singular point if $F(t_0,x_0,y_0)=0$ and

$$\frac{\partial}{\partial t}F(t_0,x_0,y_0)=\frac{\partial}{\partial x}F(t_0,x_0,y_0)=\frac{\partial}{\partial y}F(t_0,x_0,y_0)=0.$$

A singular point (t_0,x_0,y_0) of the curve $F(t,x,y)=0$ is called a double point if at least one of the second partial derivatives of $F(t,x,y)$ at (t_0,x_0,y_0) is nonzero. A double point of a curve is called an ordinary double point or a node if there are two distinct tangents at the point. We prove that the form $F(t,x,y)$ satisfying conditions (i)-(iv) admits a determinantal representation $F_S(t,x,y)$ via some cyclic weighted shift matrix $S(a_1,a_2,a_3)$ when $n=3$. For $n\geqslant 4$, we restrict our attention to the case that the curve $F(t,x,y)=0$ of order n is irreducible and the genus $g=1$, i.e., the number of real nodes of C_F is

$$\frac{(n-1)(n-2)}{2}-1=\frac{n(n-3)}{2}$$

(cf. [5]). Under this assumption, we prove that such form $F(t,x,y)$ also admits a determinantal representation $F_S(t,x,y)$ via some cyclic weighted shift matrix S when $n=4,5$. This determinantal method can be applied to the cases $n=6,7,8$ (cf. [7]). However, we can not efficiently perform this method for $n\geqslant 9$ because of the complexity on configuration of the singular points of the algebraic curve $F(t,x,y)=0$.