Applied Mathematics and Computation 258 (2015) 182–191

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Bicyclic oriented graphs with skew-rank 2 or 4

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ARTICLE INFO

Keywords:
Oriented graph
Skew-adjacency matrix
Skew-rank
Rank
Bicyclic graph

ABSTRACT

The skew-rank of oriented graph G^σ, denoted by $sr(G^\sigma)$, is the rank of the skew-adjacency matrix of G^σ. The skew-rank is even since the skew-adjacency matrix is skew-symmetric. In this paper we characterize the bicyclic oriented graphs with skew-rank 2 or 4.

1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $A(G)$ of a graph G of order n is the $n \times n$ symmetric 0–1 matrix $(a_{ij})_{n\times n}$ such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. We denote by $Sp(G)$ the spectrum of $A(G)$. The rank of $A(G)$ is called to be the rank of G, denoted by $r(G)$. Let G^σ ba a graph with an orientation which assigns to each edge of G a direction so that G^σ becomes an oriented graph. The graph G is called the *underlying graph* of G^σ. The *skew-adjacency matrix* associated to the oriented graph G^σ is defined as the $n \times n$ matrix $S(G^\sigma) = (s_{ij})$ such that $s_{ij} = 1$ if there has an arc from v_i to v_j, $s_{ij} = -1$ if there has an arc from v_j to v_i and $s_{ij} = 0$ otherwise. Obviously, the skew-adjacency matrix is skew symmetric. The *skew-rank* of an oriented graph G^σ, denoted by $sr(G^\sigma)$, is defined as the rank of the skew-adjacency matrix $S(G^\sigma)$. The *skew-spectrum* $Sp(G^\sigma)$ of G^σ is defined as the spectrum of $S(G^\sigma)$. Note that $Sp(G^\sigma)$ consists of only purely imaginary eigenvalues and the skew-rank of an oriented graph is even.

Let $C_k^\sigma = u_1, u_2, \ldots, u_k u_1$ be an even oriented cycle. The *sign* of the even cycle C_k^σ, denoted by $sgn(C_k^\sigma)$, is defined as the sign of $\prod_{i=1}^k s_{u_i u_{i+1}}$ with $u_{k+1} = u_1$. An even oriented cycle C_k^σ is called *evenly-oriented* (*oddly-oriented*) if its sign is positive (negative). If every even cycle in G^σ is evenly-oriented, then G^σ is called *evenly-oriented*. An *induced subgraph* of G^σ is an induced subgraph of G and each edge preserves the original orientation in G^σ. For an induced subgraph H^σ of G^σ, let $G^\sigma - H^\sigma$ be the subgraph obtained from G_w by deleting all vertices of H_w and all incident edges. For $V' \subseteq V(G^\sigma)$, $G^\sigma - V'$ is the subgraph obtained from G^σ by deleting all vertices in V' and all incident edges. A vertex of a graph G^σ is called *pendant* if it is only adjacent to one vertex, and is called *quasi-pendant* if it is adjacent to a pendant vertex. Denote by P_n, S_n, C_n, K_n a path, a star, a cycle and a complete graph all of which are simple unoriented graphs of order n, respectively. A graph is called *trivial* if it has one vertex and no edges. Let G be a bicyclic graph. The *base* of G, denoted by $\widehat{G}$, is the unique bicyclic subgraph of G containing no pendant vertices. Thus G can be obtained from $\widehat{G}$ by attaching trees to some vertices of $\widehat{G}$. It is well known that (see, for example [14]) there are two types of bases of bicyclic graphs, as described next.

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http://dx.doi.org/10.1016/j.amc.2015.02.014
0096-3003/

Let C_p ($p \geqslant 3$) and C_q ($q \geqslant 3$) be two cycles and $P_l = v_1, v_2, \ldots, v_l$ ($l \geqslant 1$) be a path. Assume that $v \in V(C_p)$ and $u \in V(C_q)$. Denoted by $\infty(p, l, q)$ (as depicted in Fig. 1) the graph obtained from C_p, C_q, P_l by identifying v with v_1, u with v_l, respectively. The bicyclic graph containing $\infty(p, l, q)$ as its base is called an ∞-graph.

Let P_{p+2}, P_{l+2}, P_{q+2} be three paths with $\min\{p, l, q\} \geqslant 0$ and at most one of p, l, q is 0. Denoted by $\theta(p, l, q)$ (as depicted in Fig. 1) the graph obtained from P_{p+2}, P_{l+2}, P_{q+2} by identifying the three initial vertices and terminal vertices. The bicyclic graph containing $\theta(p, l, q)$ as its base is called a θ-graph.

Recently the study of the skew-adjacency matrix of oriented graphs attracted some attentions. Cavers et al. [4] provided a paper about the skew-adjacency matrices in which authors considered the following topics: graphs whose skew-adjacency matrices are all cospectral; relations between the matching polynomial of a graph and the characteristic polynomial of its adjacency and skew-adjacency matrices; skew-spectral radii and an analogue of the Perron-Frobenius theorem; and the number of skew-adjacency matrices of a graph with distinct spectra. Anuradha and Balakrihnan [2] investigated skew spectrum of the Cartesian product of an oriented graph with an oriented Hypercube. Anuradha et al. [3] considered the skew spectrum of special bipartite graphs and solved a conjecture of Cui and Hou [8]. Hou et al. [12] gave an expression of the coefficients of the characteristic polynomial of the skew-adjacency matrix $S(G^\sigma)$. As its applications, they present new combinatorial proofs of some known results. Moreover, some families of oriented bipartite graphs with $Sp(S(G^\sigma)) = iSp(G)$ ($i = \sqrt{-1}$) were given. Gong et al. [9] investigated the coefficients of weighted oriented graphs. In addition they established recurrences for the characteristic polynomial and deduced a formula for the matching polynomial of an arbitrary weighted oriented graph. Xu [27] established a relation between the spectral radius and the skew-spectral radius. Also some results on the skew-spectral radius of an oriented graph and its oriented subgraphs were derived. As applications, a sharp upper bound of the skew-spectral radius of oriented unicyclic graphs was present. Some authors investigated the skew-energy of oriented graphs, one can refer to [1,7,13,10,19,24,28]. Actually, various of graph energies are studied, such as graph energy [11,16–18,20,21,25], matching energy [5,6,15,26]. In this paper we characterize the bicyclic oriented graphs with skew-rank 2 or 4.

2. Preliminaries

The following lemma can be derived from fundamental matrix theory.

Lemma 1.

(i) Let H^σ be an induced subgraph of G^σ. Then $sr(H^\sigma) \leqslant sr(G^\sigma)$.
(ii) Let $G^\sigma = G_1^\sigma \cup G_2^\sigma \cup \cdots \cup G_t^\sigma$, where $G_1^\sigma, G_2^\sigma, \ldots, G_t^\sigma$ are connected components of G^σ. Then $sr(G^\sigma) = \sum_{i=1}^{t} sr(G_i^\sigma)$.
(iii) Let G^σ be an oriented graph on n vertices. Then $sr(G^\sigma) = 0$ if and only if G^σ is a graph without edges (empty graph).

Lemma 2 [12,23]. *Let C_n^σ be an oriented cycle of order n. Then we have*

$$sr(C_n^\sigma) = \begin{cases} n, & C_n^\sigma \text{ is oddly-oriented}, \\ n-2, & C_n^\sigma \text{ is evenly-oriented}, \\ n-1, & \text{otherwise}. \end{cases}$$

Lemma 3 [22]. *Let T^σ be an oriented tree with matching number $\beta(T)$. Then we have*

$$sr(T^\sigma) = r(T) = 2\beta(T).$$

The following result is immediate from Lemma 3.

Lemma 4. *Let P_n^σ be an oriented path of order n. Then we have*

$$sr(P_n^\sigma) = \begin{cases} n-1, & n \text{ is odd}, \\ n, & n \text{ is even}. \end{cases}$$

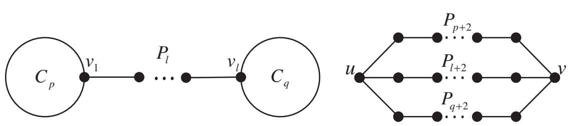

Fig. 1. $\infty(p, l, q)$ and $\theta(p, l, q)$.