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Numerical methods for the mean exit time and escape probability of two-dimensional stochastic dynamical systems with non-Gaussian noises

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ABSTRACT

The mean exit time and escape probability are deterministic quantities that can quantify dynamical behaviors of stochastic differential equations with non-Gaussian α-stable type Lévy motions. Both deterministic quantities are characterized by differential–integral equations (i.e., differential equations with nonlocal terms) but with different exterior conditions. A convergent numerical scheme is developed and validated for computing the mean exit time and escape probability for two-dimensional stochastic systems with rotationally symmetric α-stable type Lévy motions. The effects of drift, Gaussian noises, intensity of jump measure and domain sizes on the mean exit time are discussed. The difference between the one-dimensional and two-dimensional cases is also presented.

1. Introduction

Non-Gaussian stochastic dynamical systems with Lévy fluctuations have attracted a lot of attention recently, as they provide appropriate mathematical models for certain physical, geophysical, biophysical and economic phenomena [1–5]. As a significant class of non-Gaussian processes, Lévy processes describes fluctuations with features such as heavy tails and jumps. A special class of Lévy processes, α-stable Lévy processes $(0 < \alpha < 2)$, may be regarded as generalization of the well known Gaussian process, i.e., Brownian motion $(\alpha = 2)$. In general, a Lévy processes has a non-Gaussian component (whose sample paths have jumps quantified by a jump measure) and a Gaussian component (whose sample paths are continuous in time).

In this paper, we consider the following stochastic differential equation (SDE) in $\mathbb{R}^d (d = 2)$ with Lévy process

$$dX_t = b(X_t)dt + dL_t, \tag{1}$$

where the initial condition is $X_0 = x$, b is drift term (a vector field), and L_t is a Lévy process with generating triplet $(0, A, \epsilon\nu)$, where A is a symmetric non-negative definite $d \times d$ matrix, and ν is a radially symmetric Lévy jump measure on $\mathbb{R}^d \setminus \{0\}$. Our goal is to study exit problems for this stochastic system, in terms of mean exit time (how long the system remains in a bounded domain) and escape probability (likelihood of the system transition from one regime to another) [6,7]. Besides recent theoretical works in this area such as [8,9], we also see applications of mean exit time and escape probability in

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biological systems [10–12]. Higham et al. [13] presented a Monte Carlo method for the mean exit times in the context of Gaussian noise. Gao et al. [14] devised a method to compute mean exit time and escape probability in a one-dimensional stochastic system with symmetric α-stable Lévy process.

The mean exit time (MET) and escape probability (EP) are solutions to deterministic nonlocal partial differential equations. We will develop a numerical method to solve these partial differential equations with nonlocal integral terms, in the two-dimensional case ($d = 2$).

No matter what dimension is, it is difficult to deal with the nonlocal term, which is a singular integral. It is also related to the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, as seen in [15,2,16]. We consider the case where the MET and EP are radially symmetric, for example, when the domains, the drift term and the diffusion term are radially symmetric. We exploit the Gauss-Hypergeometric (G-H) function in designing our numerical method. Recently, Luan et al. [17] used a special function to make the nonlocal term easier to deal with $0 < \alpha \le 1$ in the two-dimensional case. However, the numerical method in [17] is not convergent for $\alpha > 1$, while our method converges for all values of α.

This paper is divided into several parts. In Section 2, we review several basic concepts about Lévy processes, mean exit time and escape probability. Then in Section 3, we devise a numerical method for computing mean exit time and escape probability by discretizing nonlocal partial differential equations. We will compare the numerical approximations and the analytic solutions for special cases, and find MET and EP for nonzero drift and/or diffusion cases in Section 4. We conclude this paper with some discussions in the final section.

2. Primary concepts

2.1. Lévy processes

The Lévy–Khintchine formula for a Lévy process L_t with generating triplet $(0, A, \epsilon\nu)$ dictates its characteristic function to be [15,18]

$$\mathbb{E}e^{i(u,L_t)} = \exp\left\{-\frac{1}{2}(u,Au)t + \epsilon t\int_{\mathbb{R}^d\setminus\{0\}}(e^{i(u,y)} - 1 - i(u,y)1_B(y))\nu(\mathrm{d}y)\right\},$$

where 1_B is the indicator function of the unit ball B centered at the origin. This Lévy process has a Gaussian component described by the symmetric non-negative definite matrix A, and a non-Gaussian component depicted by the Lévy jump measure ν. In the following, we mainly consider the radially symmetric Lévy jump measure for $0 < \alpha < 2$

$$\nu(dy) = \frac{C_{d,\alpha}}{|y|^{d+\alpha}}\mathrm{d}y, \quad \text{where } C_{d,\alpha} = \frac{\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{2}$$

Recall that a symmetric α-stable Lévy process, with $0 < \alpha < 2$, has triplet $(0, 0, \nu)$. Its characteristic function is

$$\mathbb{E}e^{i(u,L_t)} = e^{-Ct|u|^{\alpha}}, \quad u \in \mathbb{R}^d, \tag{3}$$

where $C = \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d+\alpha}{2})}$.

For every $\varphi \in H_0^2(\mathbb{R}^d)$, the generator for X_t is defined as [15]

$$\mathscr{L}\varphi(x) = b_i\partial_i\varphi(x) + \frac{1}{2}a_{ij}\partial_i\partial_j\varphi(x) + \epsilon C_{d,\alpha}\int_{\mathbb{R}^d\setminus\{0\}}\frac{\varphi(x+y) - \varphi(x) - 1_{B_\delta(0)}\, y_j\partial_j\varphi(x)}{|y|^{d+\alpha}}\,\mathrm{d}y, \tag{4}$$

where we have applied Einstein summation convention (i.e., any repeated index indicates summation over that index). Here we use $1_{B_\delta(0)}$ instead of $1_{B_1(0)}$because the domain $B_1(0)\setminus B_\delta(0)$ is symmetric, thus

$$\int_{B_1(0)\setminus B_\delta(0)}\frac{y_j\partial_j\varphi(x)}{|y|^{d+\alpha}}\,\mathrm{d}y = 0.$$

2.2. Mean exit time and escape probability

For SDE (1), the first exit time starting at x from a bounded domain D is defined as $\tau(\omega) := \inf\{t \geqslant 0, X_t(\omega, x) \notin D\}$, and the mean first exit time (in short, mean exit time or MET) is $u(x) = \mathbb{E}[\tau(\omega)]$.

Assume that $b_i(x)$ and $a_{ij}(x)$ satisfy a Lipschitz condition and linear growth condition (for the existence and uniqueness of solution [15]). Due to the Dynkin's formula, the mean exit time satisfy the following nonlocal (integro-differential) partial differential equation [2]

$$\mathscr{L}u(x) = -1, \quad \text{for} \quad x \in D, \tag{5}$$

subject to the Dirichlet-type exterior condition,

$$u(x) = 0, \quad \text{for} \quad x \in D^c \tag{6}$$