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Switching from exact scheme to nonstandard finite difference scheme for linear delay differential equation

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ABSTRACT

One-dimensional models are important for developing, demonstrating and testing new methods and approaches, which can be extended to more complex systems. We design for a linear delay differential equation a reliable numerical method, which consists of two time splits as follows: (a) It is an exact scheme at the early time evolution $-\tau \leqslant t \leqslant \tau$, where τ is the discrete value of the delay; (b) Thereafter, it is a nonstandard finite difference (NSFD) scheme obtained by suitable discretizations at the backtrack points. It is shown theoretically and computationally that the NSFD scheme is dynamically consistent with respect to the asymptotic stability of the trivial equilibrium solution of the continuous model. Extension of the NSFD to nonlinear epidemiological models and its good performance are tested on a numerical example.

1. Introduction

Delay differential equations are extensively used in the modeling of biological systems, specifically in the epidemiology of infectious diseases [3,7,11]. One of the popular approaches in the study of the qualitative behavior of such models is their linearization about the equilibria [3,10,11,17,18]. As a motivation, we consider the logistic delay model that arises in the modeling of communicable diseases [11], including the transmission dynamics of gonorrhea in a homosexually active population, [6]. In the latter specific case, the model is given by

$$I'(t) = \beta\left(1 - \frac{1}{\mathcal{R}_0}\right) I(t)\left(1 - \frac{I(t-\tau)}{N\left(1 - \frac{1}{\mathcal{R}_0}\right)}\right), \tag{1}$$

where β is the contact rate, $\mathcal{R}_0$ the basic reproduction number, N the total population; τ, is here and after the delay in infectivity. It is evident from (1) that when $\mathcal{R}_0 \leqslant 1$, the only equilibrium is the disease free equilibrium (DFE), $I^* = 0$. However, if $\mathcal{R}_0 > 1$, in addition to the DFE, there is an endemic equilibrium (EE), $I^{**} = N(1 - \frac{1}{\mathcal{R}_0})$. Linearizing (1) about the equilibria I^* and I^{**}, we have

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$$
\begin{aligned}
&I'(t) = \beta\left(1 - \frac{1}{\mathcal{R}_0}\right)I(t) \quad \text{and} \\
&X'(t) = -\beta\left(1 - \frac{1}{\mathcal{R}_0}\right)X(t-\tau), \quad X(t) = I(t) - N\left(1 - \frac{1}{\mathcal{R}_0}\right),
\end{aligned}
\tag{2}
$$

respectively. More generally, if the spread of a disease is modeled by a system of delay differential equations, such as in [10,17,18], then linearization about the equilibrium points, with the Jacobian matrix assumed to be diagonizable, leads to linearized delay differential equations of the type in (2). Thus, the general setting of this work is a linear delay differential equation (LDDE),

$$
\begin{aligned}
&x'(t) = Ax(t) + Bx(t-\tau) + f(t) \quad t > 0, \\
&x(t) = \phi(t) \quad t \in [-\tau, 0],
\end{aligned}
\tag{3}
$$

where A and B are constants, while $f : [0, +\infty) \to \mathbb{R}$ and $\phi : [-\tau, 0] \to \mathbb{R}$ are continuous functions, with ϕ being the initial function.

The well-posedness of LDDE (3) can be stated as follows [9]:

Theorem 1. *Under the assumptions stated above, there exists a unique continuously differentiable function* $x : [-\tau, +\infty) \to \mathbb{R}$ *which solves LDDE* (3). *The solution is represented by the Volterra integral equation*

$$
\begin{aligned}
&x(t) = \phi(t), \quad t \in [-\tau, 0], \\
&x(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)}[Bx(s-\tau) + f(s)]ds, \quad t \geqslant 0.
\end{aligned}
$$

Regarding the qualitative feature of (3), we consider the homogeneous equation

$$
x'(t) = Ax(t) + Bx(t-\tau),
\tag{4}
$$

in which we assume without loss of generality that $A + B \neq 0$ so that $x = 0$ is the only equilibrium point of (4). The characteristic equation of (4) is the following transcendental function of the complex argument λ:

$$
\lambda - A - Be^{-\lambda\tau} = 0.
\tag{5}
$$

We have the following stability result ([5], Theorem 13.8):

Theorem 2. *The equilibrium $x = 0$ is asymptotically stable, or equivalently, all roots of (5) have their real parts strictly less than zero if, and only if, the following two conditions hold:*

(a) $A < 1/\tau$;

(b) $A < -B < \sqrt{(a_1/\tau)^2 + A^2}$ *where a_1 is the root of the equation $a = A\tan a$ with $0 < a_1 < \pi, a \in \mathbb{R}$, on the understanding that $a_1 = \pi/2$ if $A = 0$.*

In the absence of delay ($\tau = 0$) and if $f \equiv 0$, Equation LDDE (3) reduces to

$$
x'(t) = (A + B)x(t),
\tag{6}
$$

Eq. (6) is the well-known exponential equation, which is of paramount importance from both the theoretical and numerical analysis point of view in the study of dynamical systems, without delay, of the form

$$
x'(t) = g(x), \quad g(0) = 0.
\tag{7}
$$

The relevance of (6) from the constructive point of view hinges on the explicit and implicit knowledge of its exact scheme, which is [16],

$$
\frac{x_{n+1} - x_n}{(\exp[(A+B)\Delta t] - 1)/(A+B)} = (A+B)x_n,
\tag{8}
$$

or

$$
\frac{x_{n+1} - x_n}{[1 - \exp(-(A+B)\Delta t)]/(A+B)} = (A+B)x_{n+1},
\tag{9}
$$

where x_n denotes here and after an approximation of the solution $x(t)$ at the discrete time $t_n = n\Delta t, n = 0, 1, 2, \ldots, \Delta t$ being the time step size. Most reliable nonstandard finite difference (NSFD) schemes for Eq. (7) are designed on the basis of the exact scheme (8) or (9), assuming that (6) is the linearized equation of (7) about the trivial equilibrium.

The purpose of this work is to design reliable NSFD schemes for LDDE (3). The ideal situation is to produce its exact scheme. According to Theorem 1.1 in [16], an exact scheme is readily determined once the solution of the continuous differential model is known. However, this theorem does not apply here because the second formula in Theorem 1 is an integral