Applied Mathematics and Computation 258 (2015) 483–488

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Exponential stability of a second order delay differential equation without damping term

Leonid Berezansky [a,*], Alexander Domoshnitsky [b], Mikhail Gitman [c], Valery Stolbov [c]

[a] Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
[b] Department of Mathematics and Computer Science, Ariel University, Ariel, Israel
[c] Department of Mathematical Modeling, Perm National Research Polytechnic University, Perm, Russia

ARTICLE INFO

Keywords:
Exponential stability
Second order delay equations
Without damping

ABSTRACT

For the delay differential equation

$$\ddot{x}(t) + a(t)x(h(t)) - b(t)x(g(t)) = 0, \quad g(t) \leqslant t, \quad h(t) \leqslant t,$$

without damping term, explicit exponential stability conditions are obtained.

1. Introduction

It is well-known, that introducing delays in the first order ordinary differential equation (ODE) $\dot{x}(t) + ax(t) = 0$ may destroy asymptotic stability but not vice versa. For example, although this equation is asymptotically stable for all $a > 0$, the equation $\dot{x}(t) + ax(t-h) = 0$ is asymptotically stable iff $0 < ah < \pi/2$. Some other specific properties of linear delay differential equations of the first order one can find in [8,9]. For second order differential equations we have a similar situation. The general solution of the ODE $x''(t) + ax(t) = 0 \ \ (a > 0)$ is $x(t) = c_1 \sin\sqrt{a}t + c_2 \cos\sqrt{a}t$, so it is bounded. Myshkis [14] proved that for all pairs of positive constants a and τ, there exists unbounded solution to the delay equation $x''(t) + ax(t-\tau) = 0$. Assertions on unboundedness of solutions for linear delay equation with variable and bounded coefficients and delays $x''(t) + \sum_{i=1}^{m} a_i(t)x(t-\tau_i(t)) = 0$, which extend this result by Myshkis, were obtained in [10]. In the case of a nondecreasing positive and bounded coefficient $a(t)$, the inequality $\int^{\infty}\tau(t)dt < \infty$ is necessary and sufficient for boundedness of all solutions to the equation $x''(t) + a(t)x(t-\tau(t)) = 0$ on the semiaxis [10]. Results on the exponential stability of second order delay differential equations without damping term were considered many years as impossible.

In [7] the authors considered the equation

$$\ddot{x}(t) = q_1 x(t) + q_2 x(t-\tau), \tag{1.1}$$

where $q_1 < 0, q_2 > 0, \tau > 0$ and obtained the following result.

Theorem 1 [7]. *Denote $B = \tau^2 q_1, D = \tau^2 q_2$. Eq. (1.1) is asymptotically stable iff there exists $k \in N$ such that*

$$2k\pi < \sqrt{-B} < (2k+1)\pi, \quad D < \min\left\{-(2k)^2\pi^2 - B, (2k+1)^2\pi^2 + B\right\}.$$

* Corresponding author.
E-mail address: brznsky@cs.bgu.ac.il (L. Berezansky).

http://dx.doi.org/10.1016/j.amc.2015.01.114
0096-3003/

Note that the ODE $\ddot{x}(t) = (q_1 + q_2)x(t)$ is not asymptotically stable. In the recent paper [12] the first explicit conditions of nonoscillation and exponential stability were obtained for the delay equation with variable coefficients and delays

$$\ddot{x}(t) + a(t)x(h(t)) - b(t)x(g(t)) = 0, \quad a(t) \geqslant 0, \quad b(t) \geqslant 0, \tag{1.2}$$

and without the condition $h(t) \equiv t$ for $t \in [0, \infty)$. Note that in [12] results on the exponential stability of the equation

$$\ddot{x}(t) + \sum_{j=1}^{n} b_j(t)x_j(t - \tau_j(t)) = 0, \quad t \in [0, +\infty),$$

were also obtained. Results on the exponential stability for delay equation with damping term can be found, for example, in the works [4–6,11,13].

In this paper we will obtain exponential stability conditions for nonautonomous delay differential Eq. (1.2) of the second order with positive and negative coefficients and for some generalizations of this equation.

Note that delay differential equations of the second order have numerous applications in many sciences (see, for example, the monograph [13]). Results of this paper can be applied for stabilization of unstable equation $\ddot{x}(t) + a(t)x(h(t)) = 0$ by the help of the control delay term $b(t)x(g(t))$.

2. Main results

Consider first a general scalar second order delay differential equation

$$\ddot{x}(t) + \sum_{k=1}^{m} A_k(t)\dot{x}(G_k(t)) + \sum_{k=1}^{l} B_k(t)x(H_k(t)) = f(t), \tag{2.1}$$

under the following conditions:

(a1) A_k, B_k, f are Lebesgue measurable and essentially bounded functions on $[0, \infty)$.
(a2) $H_k, G_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $H_k(t) \leqslant t, G_k(t) \leqslant t, t \geqslant 0$.

Together with (2.1) consider for each $t_0 \geqslant 0$ an initial value problem

$$x(t) = \varphi(t), \quad \dot{x}(t) = \theta(t), \quad t < t_0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = x'_0. \tag{2.2}$$

We also assume that the following hypothesis holds

(a3) $\varphi, \theta : (-\infty, t_0) \to \mathbb{R}$, are Borel measurable bounded functions.

We will assume that conditions (a1)–(a3) hold for any equation considered in this paper.

Definition 1. A function $x : \mathbb{R} \to \mathbb{R}$ is called a solution of problem (2.1) and (2.2) if $\dot{x}$ is locally absolutely continuous on $t \in [t_0, \infty), x$ satisfies Eq. (2.1) for almost every $t \in [t_0, \infty)$ and equalities (2.2) for $t \leqslant t_0$.

Definition 2. Eq. (2.1) is uniformly exponentially stable, if there exist $M > 0, \mu > 0$, such that the solution of problem (1.2) and (2.2) with $f \equiv 0$ satisfies the estimate

$$\max\{|\dot{x}(t)|, |x(t)|\} \leqslant Me^{-\mu(t-t_0)}\left[|x_0| + |x'_0| + \sup_{s<t_0}|\varphi(s)| + \sup_{s<t_0}|\theta(s)|\right], \quad t \geqslant t_0,$$

where M and μ do not depend on t_0.

One of the main tools in the study of the exponential stability is the following Bohl–Perron theorem.

Let us introduce some functional spaces on a semi-axis. Denote by $\mathbf{L}_\infty[t_0, \infty)$ the space of scalar functions which are essentially bounded on $[t_0, \infty)$ and by $\mathbf{C}[t_0, \infty)$ the space of scalar functions which are bounded and continuous on $[t_0, \infty)$ with the supremum norm.

Lemma 1 [1]. *Suppose there exists $t_0 \geqslant 0$ such that for every $f \in \mathbf{L}_\infty[t_0, \infty)$ both the solution x of the problem (2.1) with initial condition*

$$x(t) = 0, \quad \dot{x}(t) = 0, \quad t \leqslant t_0, \tag{2.3}$$

and its derivative $\dot{x}$ belong to $\mathbf{C}[t_0, \infty)$. Then Eq. (2.1) is uniformly exponentially stable.

If Eq. (2.1) is uniformly exponentially stable then for every $f \in \mathbf{L}_\infty[t_0, \infty)$ both the solution x of the problem (2.1) with initial condition (2.3) and its derivative $\dot{x}$ belong to $\mathbf{C}[t_0, \infty)$.